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ÚSTAV MATEMATIKY

MODEL WITH WEIBULL RESPONSES

MODELY S WEIBULLOVÝM ROZDĚLENÍM

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

AUTHOR
AUTOR PRÁCE
Bc. Tereza Konečná

SUPERVISOR
VEDOUCÍ PRÁCE
doc. Mgr. Zuzana Hübnerová, Ph.D.

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Department:	Institute of Mathematics
Student:	**Bc. Tereza Konečná**
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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Model with Weibull responses

Concise characteristic of the task:

Weibull distribution is widely applied in modelling time to a failure of a technical device. Therefore the thesis aims on regression models, where the response variables have Weibull distribution.

Goals Master's Thesis:

- Description of the properties of Weibull distribution.
- List of used terms.
- Derivation of parameter estimation methods in models with Weibull distribution with special attention to the one-way ANOVA type models.
- Description of the tests of hypothesis on parameters in Weibull models.
- Application of obtained results on simulated or real data analysis.

List of literature:

DOBSON, Annette J. and Adrian G. BARNETT. An introduction io generalized linear models. 2nd ed. Boca Raton: CRC Press, c2002. Texts in statistical science. ISBN 1-58488-165-8

MURTHY, D. N. Prabhakar, XIE, Min and JIANG, Renyan. Weibull Models. Hoboken: John Wiley & Sons, 2004. ISBN 0-471-36092-9.

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In Brno,

L. S.

prof. RNDr. Josef Šlapal, CSc.	doc. Ing. Jaroslav Katolický, Ph.D.
Director of the Institute	FME dean

Summary

This Master's thesis deals with the Weibull model, exactly the two-parametric Weibull distribution. The thesis deals with the estimation of parameters by four way of method of quantiles, by method of maximum likelihood and by graphical method Weibull probability plot. The derivation of parameter estimation methods in the oneway ANOVA type models with Weibull distribution was presented. Relations for the model with constant scale parameter α, constant shape parameter β and the model with both parameters constant were derived. Also the tests with nuisance parameters are included, namely the score test, the Wald test, and the likelihood ratio test.

The last chapter deals with the applications of the methods. A comparison of the different methods are demonstrated by graphs, histograms and tables. The methods are programmed in freeware R software. The functionality and properties of each method are verified on two sets of simulated data. In the end of the chapter tree simulated random samples are analysed.

Keywords

Weibull distribution, estimation of parameter, method of quantiles, maximum likelihood estimation, one-way ANOVA

Abstrakt

Tato diplomová práce se zabývá Weibullovými modely, přesněji dvouparametrickým Weibullovým rozdělením. Práce se zabývá odhady parametrů, a to čtyřmi variantami kvantilové metody, metodou maximální věrohodnosti a grafickou metodou Weibullova pravděpodobnostního grafu. Je uvedeno odvození odhadu parametrů pro jednovýběrovou analýzu rozptylu pro Weibullovo rozdělení. Jsou zde odvozeny vztahy pro model s konstantním parametrem α, s konstantním parametrem β a s oběma konstantními parametry. Také jsou uvedeny testové statistiky pro rušivé parametry skórový test, Waldův test a test založený na věrohodnostním poměru.

V poslední kapitole je provedena aplikace jednotlivých představených metod. Srovnání metod je ukázáno pomocí grafů, histogramů a tabulek. Metody jsou naprogramovány v softwaru R. Jejich funkčnost a vlastnosti jsme ověřili na dvou simulovaných souborech dat. Diplomová práce je zakončena příkladem tří simulovaných náhodných výběrů, na kterých byla provedena analýza pomocí zavedených metod.

Klíčová slova

Weibullovo rozdělení, odhad parametrů, metoda kvantilů, odhad maximální věrohodnosti, jednovýběrová ANOVA

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I hereby certify that this thesis is the result of my own work and I have properly cited all sources used in the thesis.

Bc. Tereza Konečná

I would like to express my gratitude to doc. Mgr. Zuzana Hübnerová, Ph.D. for the great help and all the useful tips which made this thesis as complete as it is presented.

Bc. Tereza Konečná

Contents

Introduction

This Master's thesis Model with Weibull responses introduces a description of the properties of Weibull distribution and several parameter estimation methods.

Weibull distribution was presented by professor Waloddi Weibull in 1951. Together with the normal, exponential, t^2-, $F-$ and χ^2-distributions the Weibull distribution is the most popular model in modern statistic. It has ability to fit to data from various fields, ranging from life data to observations made in economics and business administration or weather data, in biology, in hydrology or in the engineering sciences.

Chapter 0 presents the theory that will be used later in the next chapters.

Chapter 1 presents the general characteristics of Weibull distribution and specifically for the two-parameter Weibull distribution.

Chapter 2 deals with the estimation of parameters. There are the methods of Quantiles for 4 different ways of choice the quantiles considered. Furthermore, the maximum probability method is discussed. A representative of graphical methods for estimating parameters is the method of least squares applied in Weibull probability plot. All of the above mentioned methods are programmed in the enclosed program in the R software.

Chapter 3 describes the tests of hypotheses that a random sample comes from a Weibull distribution. The goodness of fit of the Weibull distribution can be assessed by the test of χ^2-type and a test based on EDF statistics, namely the Kolmogorov-Smirnov statistic and the Anderson-Darling statistics. The critical values for these test are give for both cases - when the distribution is fully specified and under the composite hypothesis.

Chapter 4 presents the derivation of parameter estimation methods in the one-way ANOVA type models with Weibull distribution. Relations are derived for the model with constant parameter scale α, the model with constant parameter shape β and the model with both parameters constant. Comparison of different types of methods follows. Included are also tests with nuisance parameters, namely the Test Score, the Wald test and the Likelihood ratio test.

Chapter 5 deals with the applications of the individual methods. Follows a comparison of the different types of methods. The methods are programmed in freeware R software. The functionality and properties of each method are verified on two sets of simulated data.

Chapter 0

Theoretical Basis

Here we introduce the terms used in the coming text.

0.1 Gamma Function

The *complete Gamma function* is given by

$$\Gamma(k) = \int_0^\infty x^{k-1} \exp[-x] \, \mathrm{d}x, \; k > 0. \tag{1}$$

For $k > 0$, the following is true

$$\Gamma(k+1) = k\Gamma(k), \tag{2}$$

$$\Gamma'(k) = \int_0^\infty x^{k-1} \exp[-x] \ln x \, \mathrm{d}x, \tag{3}$$

$$\Gamma''(k) = \int_0^\infty x^{k-1} \exp[-x] \ln^2 x \, \mathrm{d}x. \tag{4}$$

These formulas are used for the calculation in next parts.

0.2 Polygamma Function

The primary source of this section is the book [1].

The *polygamma function of order* n is given by the formula

$$\psi^{(n)}(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} \ln \Gamma(z) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1 - e^{-t}} \, \mathrm{d}t \tag{5}$$

for $z > 0$ and $n = 0, 1, \ldots$

For $n = 0$, the polygamma function has the form of

$$\psi^{(0)}(z) = \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{6}$$

The polygamma function is used to computing the derivative of the Gamma function. From (6) the following holds

$$\Gamma'(z) = \psi^{(0)}(z) \Gamma(z) \tag{7}$$

and after derivation we have the second derivation of the Gamma function as

$$\Gamma''(z) = \psi^{(1)}(z) \Gamma(z) + \psi^{(0)}(z) \Gamma'(z). \tag{8}$$

0.3 Estimation of Distribution Functions

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from distribution with distribution function $F_0(x, \boldsymbol{\theta})$ where $\boldsymbol{\theta}$ is from the parametric space Ω.

Let $X_{(1),n}, X_{(2),n}, \ldots, X_{(n),n}$ be the ordered random sample, which means that

$$X_{(1),n} \leq X_{(2),n} \leq \ldots \leq X_{(n),n}.$$

The *empirical distribution function* (EDF) is a step function calculated from the sample of size n with distribution function $F_0(x, \boldsymbol{\theta})$, which estimate a distribution function $F_0(x, \boldsymbol{\theta})$. The EDF is defined by

$$\hat{F}_n(x) = \begin{cases} 0 & \text{for } x < X_{(1),n}, \\ \frac{i}{n} & \text{for } X_{(i),n} \leq x < X_{(i+1),n}, i = 1, ..., n-1, \\ 1 & \text{for } x \geq X_{(n),n}. \end{cases} \tag{9}$$

The EDF $\hat{F}_n$ is the best unbiased estimation of distribution function F_0 [2].

In the article [5] the *modified distribution function* is used its form of

$$\hat{F}_n^*(x) = \begin{cases} 0 & \text{for } x < X_{(1),n}, \\ \frac{i-0{,}3}{n-0{,}4} & \text{for } X_{(i),n} \leq x < X_{(i+1),n}, i = 1, ..., n-1, \\ 1 & \text{for } x \geq X_{(n),n}. \end{cases} \tag{10}$$

According to the article [5], this form is the most superior for two-parameter Weibull distribution, which was verified by simulations and by comparison with other estimates.

For example, in the paper [5] has been shown that the form $\hat{F}_n(x)$ should not be used for estimation of parameters. This is one of the reasons why we used the modified distribution function in the estimation of parameters (2.45) and (2.46).

0.4 Estimator of Quantile Function

All sample quantiles are defined as weighted averages of consecutive order statistics. In the article [15] the sample quantiles are defined by:

$$\hat{Q}(p) = (1-\gamma)X_{(j)} + \gamma X_{(j+1)},$$

where $\frac{j}{n+1} \leq p < \frac{j+1}{n+1}$, $X_{(j)}$ is the jth ordered random sample, n is the sample size, the value of γ is a function of $j = \lfloor p(n+1) \rfloor$ and $\gamma = p(n+1) - j$, where p we can be computed as $p_i = \frac{i}{n+1}$ for $i = 1, \ldots, n$. Thus $p_i = E[F(X_{(i)})]$. This way of computing sample quantile is used in softwares Minitab or R. More information we can be found in the article [6].

The sample quantiles can be obtained equivalently by linear interpolation between the points $(p_k, X_{(k)})$.

Another possible option for the estimation of quantile function is that we define $\hat{Q}(p)$ such that

$$\hat{Q}(p) = \min\{x; \hat{F}_n(x) \geq p\}, \tag{11}$$

where $p \in [0; 1]$. This way of estimation has been taken from the book [2].

Chapter 1

Weibull distribution

The primary source of first chapter was the book [3].

1.1 Three-parameter Weibull distribution

The tree-parameter Weibull distribution function is characterized by the distribution function which has a form of

$$F(x;\alpha,\beta,\tau) = \begin{cases} 1 - \exp\left[-\left(\frac{x-\tau}{\alpha}\right)^{\beta}\right], & x \geq \tau, \\ 0, & x < \tau, \end{cases} \tag{1.1}$$

$\alpha > 0, \beta > 0, \tau \geq 0$. The parameter α is named the *scale*, β is the *shape* and τ is called the *location parameter*.

1.2 Two-parameter Weibull distribution

The two-parameter Weibull distribution is a special case of (1.1). It is very often called *standard Weibull distribution*. Its distribution function has the form

$$F(x;\alpha,\beta) = \begin{cases} 1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right], & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{1.2}$$

$\alpha > 0, \beta > 0$.

The *density function* is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{\beta x^{\beta-1}}{\alpha^{\beta}} \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right], & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{1.3}$$

$\alpha > 0, \beta > 0$.

The *moment generating function* is given by

$$\psi\left(r\right) = \alpha^{r}\Gamma\left(1 + \frac{r}{\beta}\right). \tag{1.4}$$

The moment generating function can be easily derived as the expected value of X^r where using the formulas from section 0.1.

$$\psi\left(r\right) = E\left[X^{r}\right] = \int_{0}^{\infty} x^{r} f(x;\alpha,\beta)\,\mathrm{d}x$$

After substitution of the theoretical density function by the density function for two-parametric Weibull model we use the substitution $u = \left(\frac{x}{\alpha}\right)^{\beta}$

$$\int_0^{\infty} x^r f(x;\alpha,\beta)\,\mathrm{d}x = \int_0^{\infty} x^r \frac{\beta x^{\beta-1}}{\alpha^{\beta}} \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\mathrm{d}x = \left| \begin{array}{l} u = \left(\frac{x}{\alpha}\right)^{\beta} \\ \mathrm{d}u = \frac{\beta x^{\beta-1}}{\alpha^{\beta}}\,\mathrm{d}x \end{array} \right| =$$
$$= \int_0^{\infty} u^{\frac{r}{\beta}} \alpha^r \frac{\beta x^{\beta-1}}{\alpha^{\beta}} \exp\left[-u\right] \frac{\alpha^{\beta}}{\beta x^{\beta-1}}\,\mathrm{d}u = \alpha^r \int_0^{\infty} u^{\frac{r}{\beta}} \exp\left[-u\right]\,\mathrm{d}u$$

Next we use (1) and then we derive the form of the moment generating function (1.4).

$$\alpha^r \int_0^{\infty} u^{\frac{r}{\beta}} \exp\left[-u\right]\,\mathrm{d}u = \alpha^r \Gamma\left(1+\frac{r}{\beta}\right)$$

In the next derivations we use the same principle as in the previous derivation. We also need (3) and (4).

$$E\left[X^r \ln X\right] = \int_0^{\infty} x^r \ln x f(x;\alpha,\beta)\,\mathrm{d}x = \int_0^{\infty} x^r \ln x \frac{\beta x^{\beta-1}}{\alpha^{\beta}} \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\mathrm{d}x =$$
$$= \left| \begin{array}{l} u = \left(\frac{x}{\alpha}\right)^{\beta} \\ \mathrm{d}u = \frac{\beta x^{\beta-1}}{\alpha^{\beta}}\,\mathrm{d}x \end{array} \right| = \int_0^{\infty} u^{\frac{r}{\beta}} \alpha^r \left(\frac{1}{\beta}\ln u + \ln\alpha\right) \frac{\beta x^{\beta-1}}{\alpha^{\beta}} \exp\left[-u\right] \frac{\alpha^{\beta}}{\beta x^{\beta-1}}\,\mathrm{d}u =$$
$$= \frac{\alpha^r}{\beta} \int_0^{\infty} u^{\frac{r}{\beta}} \ln u \exp\left[-u\right]\,\mathrm{d}u + \alpha^r \ln\alpha \int_0^{\infty} u^{\frac{r}{\beta}} \exp\left[-u\right]\,\mathrm{d}u =$$
$$= \frac{\alpha^r}{\beta}\Gamma'\left(1+\frac{r}{\beta}\right) + \alpha^r \ln\alpha\,\Gamma\left(1+\frac{r}{\beta}\right) \tag{1.5}$$

$$E\left[X^r \left(\ln X\right)^2\right] = \int_0^{\infty} x^r \left(\ln x\right)^2 f(x;\alpha,\beta)\,\mathrm{d}x = \int_0^{\infty} x^r \left(\ln x\right)^2 \frac{\beta x^{\beta-1}}{\alpha^{\beta}} \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\mathrm{d}x$$
$$= \left| \begin{array}{l} u = \left(\frac{x}{\alpha}\right)^{\beta} \\ \mathrm{d}u = \frac{\beta x^{\beta-1}}{\alpha^{\beta}}\,\mathrm{d}x \end{array} \right| = \int_0^{\infty} u^{\frac{r}{\beta}} \alpha^r \left(\frac{1}{\beta}\ln u + \ln\alpha\right)^2 \frac{\beta x^{\beta-1}}{\alpha^{\beta}} \exp\left[-u\right] \frac{\alpha^{\beta}}{\beta x^{\beta-1}}\,\mathrm{d}u$$
$$= \alpha^r \int_0^{\infty} u^{\frac{r}{\beta}} \left(\frac{1}{\beta}\ln u + \ln\alpha\right)^2 \exp\left[-u\right]\,\mathrm{d}u$$
$$= \frac{\alpha^r}{\beta^2} \int_0^{\infty} u^{\frac{r}{\beta}} \left(\ln u\right)^2 \exp\left[-u\right]\,\mathrm{d}u + \frac{2\alpha^r}{\beta} \int_0^{\infty} + u^{\frac{r}{\beta}} \ln u \ln\alpha \exp\left[-u\right]\,\mathrm{d}u$$
$$+ \alpha^r \left(\ln\alpha\right)^2 \int_0^{\infty} u^{\frac{r}{\beta}} \exp\left[-u\right]\,\mathrm{d}u$$
$$= \frac{\alpha^r}{\beta^2}\Gamma''\left(1+\frac{r}{\beta}\right) + \frac{2\alpha^r}{\beta}\ln\alpha\,\Gamma'\left(1+\frac{r}{\beta}\right) + \alpha^r \left(\ln\alpha\right)^2 \Gamma\left(1+\frac{r}{\beta}\right) \tag{1.6}$$

The *expected value* (the first moment) is given by

$$\mu = E\left[X\right] = \alpha\Gamma\left(1+\frac{1}{\beta}\right), \tag{1.7}$$

which can be derived from (1.4).

The *variance* is given by

$$\sigma^2 = E\left[X^2\right] - \left(E\left[X\right]\right)^2 = \alpha^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right]. \tag{1.8}$$

The *quantile function* (sometimes called the *percentile function*) is given by

$$Q(p, \alpha, \beta) = \alpha \left[-\ln(1-p) \right]^{\frac{1}{\beta}}, \tag{1.9}$$

for $p \in [0; 1]$ which we can compute as the inverse function of (1.2).

Chapter 2

Parameter Estimation

In this chapter we discuss methods for estimation of parameters of the two-parameter Weibull distribution. Namely the method of quantiles, the maximum likelihood method and least square methods apply to Weibull probability plot. There also exist other methods, for example the minimum chi-square estimation which we can find in the book [2]. The maximum likelihood methods is iterative in this case and therefore we must used the method of quantiles for initial value for iteration.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the two-parameter Weibull distribution.

2.1 Method of Quantiles

In this section we follow [9]. In the method of percentiles we use the percentile function from (1.9).

The method of quantiles is based on the comparison of theoretical and empirical quantiles. Let $X_{p_i,n}$ be the sample quantile corresponding to the given probability p_i, $i = 1, ..., m$, where m is the number of parameters. The estimates of the parameters $\theta_1, ..., \theta_m$ are the solution of the system of equations

$$\begin{cases} X_{p_1,n} = Q(p_1, \theta_1, ..., \theta_m) \\ X_{p_2,n} = Q(p_2, \theta_1, ..., \theta_m) \\ \vdots \\ X_{p_m,n} = Q(p_s, \theta_1, ..., \theta_m). \end{cases} \tag{2.1}$$

For the two-parametric Weibull model ($m = 2$) there are several options to choose the quantiles p_1 and p_2. The following options are considered in the following subsections.

2.1.1 General Case

The source [4] focus on the general case of the method of quantiles. Namely

$$0 < p_1 < p_2 < 1.$$

The system of equations (2.1) has the form

$$\begin{cases} X_{p_1,n} = \alpha \left[-\ln(1 - p_1)\right]^{1/\beta} \\ X_{p_2,n} = \alpha \left[-\ln(1 - p_2)\right]^{1/\beta}. \end{cases} \tag{2.2}$$

Taking logarithms of both equations (2.2) we obtain

$$\begin{aligned} \ln X_{p_1,n} &= \ln\alpha + \tfrac{1}{\beta}\ln\left[-\ln\left(1-p_1\right)\right] \\ \ln X_{p_2,n} &= \ln\alpha + \tfrac{1}{\beta}\ln\left[-\ln\left(1-p_2\right)\right] \end{aligned} \tag{2.3}$$

We derive the difference between the two previous equations (2.3)

$$\ln X_{p_1,n} - \ln X_{p_2,n} = \frac{1}{\beta}\ln\left[-\ln\left(1-p_1\right)\right] - \frac{1}{\beta}\ln\left[-\ln\left(1-p_2\right)\right]$$

and from the difference we derive the equation for parameter β. Therefore the estimation of β has the form of

$$\hat{\beta} = \frac{\ln\left[-\ln\left(1-p_1\right)\right] - \ln\left[-\ln\left(1-p_2\right)\right]}{\ln X_{p_1,n} - \ln X_{p_2,n}} \tag{2.4}$$

We derive the estimation of parameter α from the sum of the equations (2.3)

$$\ln X_{p_1,n} + \ln X_{p_2,n} = 2\ln\alpha + \frac{1}{\beta}\ln\left[-\ln\left(1-p_1\right)\right] + \frac{1}{\beta}\ln\left[-\ln\left(1-p_2\right)\right]$$

$$\hat{\alpha} = \exp\left(\frac{1}{2}\left[\ln X_{p_1,n} + \ln X_{p_2,n} - \frac{1}{\beta}\ln\left[-\ln\left(1-p_1\right)\right] - \frac{1}{\beta}\ln\left[-\ln\left(1-p_2\right)\right]\right]\right) \tag{2.5}$$

$$= \exp\left(\frac{1}{2}\sum_{i=1}^{2}\ln\frac{X_{p_i,n}}{\left[-\ln\left(1-p_i\right)\right]^{\frac{1}{\beta}}}\right) \tag{2.6}$$

$$= \prod_{i=1}^{2}\left(\frac{X_{p_i,n}}{\left[-\ln\left(1-p_i\right)\right]^{\frac{1}{\beta}}}\right)^{\frac{1}{2}} \tag{2.7}$$

If we want to estimate the parameter α by (2.7), then we must use the estimation $\hat{\beta}$ instead of the β parameter.

We substituted (2.4) into (2.5) then we have

$$\begin{aligned} \hat{\alpha} &= \exp\left(\frac{1}{2}\left[\ln X_{p_1,n} + \ln X_{p_2,n} - \frac{1}{\beta}\ln\left[-\ln\left(1-p_1\right)\right] - \frac{1}{\beta}\ln\left[-\ln\left(1-p_2\right)\right]\right]\right) \\ &= \exp\left(\frac{1}{2}\left[\ln X_{p_1,n} + \ln X_{p_2,n} - \frac{\ln X_{p_1,n} - \ln X_{p_2,n}}{k}\ln k_1 - \frac{\ln X_{p_1,n} - \ln X_{p_2,n}}{k}\ln k_2\right]\right) \\ &= \exp\left(\frac{1}{2}\left[\ln X_{p_1,n} + \ln X_{p_2,n} + \frac{\left(\ln X_{p_1,n} - \ln X_{p_2,n}\right)\left(\ln k_1 - \ln k_2 - 2\ln k_1\right)}{k}\right]\right) \\ &= \exp\left(\frac{1}{2}\left[2\ln X_{p_1,n} - 2\frac{\left(\ln X_{p_1,n} - \ln X_{p_2,n}\right)\ln k_1}{k}\right]\right) \\ &= \exp\left(\frac{1}{2}\left[2\ln X_{p_1,n} - 2\frac{\ln X_{p_1,n}\ln k_1}{k} + 2\frac{\ln X_{p_2,n}\ln k_1}{k}\right]\right) \\ &= \exp\left(\ln X_{p_1,n} - \frac{\ln k_1}{k}\ln X_{p_1,n} + \frac{\ln k_1}{k}\ln X_{p_2,n}\right) \\ &= \exp\left(w\ln X_{p_1,n} + (1-w)\ln X_{p_2,n}\right), \end{aligned}$$

where

$$k = \ln\left[-\ln\left(1-p_1\right)\right] - \ln\left[-\ln\left(1-p_2\right)\right],\ k_i = -\ln\left(1-p_i\right),\ w = 1 - \frac{\ln k_1}{k} \tag{2.8}$$

for $i = 1, 2$.

Let as consider the asymptotic distribution of the estimate of parameters α and β by the method of quantiles as in [4]. This well be done by using the results of [8] and [7].

Asymptotic Covariance Matrix

The primary source are the articles [8], [7] and [4].

Theorem 2.1. *[8] Let the probability density function $f(x)$ sutisfy that is continuous, and does not vanish in the neighborhood of $X_{p_i,n}$, where*

$$\int_{-\infty}^{X_{p_i,n}} f(x)\,\mathrm{d}x = p_i$$

for $i = 1,\ldots,m$. Then $X_{p_i,n}$ are asymptotically distributed according to the normal multivariate distribution, with mean $Q(p_i,\alpha,\beta)$ variance

$$\sigma_i^2 = \frac{p_i(1-p_i)}{nf^2(Q(p_i,\alpha,\beta))} \tag{2.9}$$

for $i = 1,\ldots,m$ and covariance

$$\rho_{ij}\sigma_i\sigma_j = \frac{p_i(1-p_j)}{nf(Q(p_i,\alpha,\beta))\,f(Q(p_j,\alpha,\beta))} \tag{2.10}$$

for $1 \le i < j \le m$.

Theorem 2.2. *[7] Suppose that*

$$\sqrt{n}\,(\mathbf{X}-\boldsymbol{\theta}),$$

where $\mathbf{X} = (X_1,\ldots,X_n)$, has asympthotic multivariate normal distribution with mean vector $\boldsymbol{0}$ and covariance matrix $\boldsymbol{\Sigma}$, and suppose that $h_1,\ldots,h_m$ are m real-valued functions of $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_m)$, defined and continuously differentiable in a neighborhood ω of the parameter point θ and such that the matrix $\mathbf{B} = \left\{\frac{\partial h_i}{\partial \theta_j}\right\}_{i,j=1,\ldots,m}$ of partial derivatives is nonsingular in ω. Then,

$$\left[\sqrt{n}\,[h_1(\mathbf{X}) - h_1(\boldsymbol{\theta})],\ldots,\sqrt{n}\,[h_m(\mathbf{X}) - h_m(\boldsymbol{\theta})]\right]$$

has asympthotic multivariate normal distribution with mean vector $\boldsymbol{0}$ and with covariance matrix

$$\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^T.$$

The function h_1 and the first derivative of function h_1 have the form

$$h_1 = \alpha = \exp(w\ln Q_1 + (1-w)\ln Q_2) \tag{2.11}$$

$$\frac{\mathrm{d}h_1}{\mathrm{d}Q_1} = \exp(w\ln Q_1 + (1-w)\ln Q_2)\frac{w}{Q_1} = \alpha\frac{w}{\alpha k_1^{1/\beta}} = \frac{w}{k_1^{1/\beta}} \tag{2.12}$$

$$\frac{\mathrm{d}h_1}{\mathrm{d}Q_2} = \exp(w\ln Q_1 + (1-w)\ln Q_2)\frac{1-w}{Q_2} = \alpha\frac{1-w}{\alpha k_2^{1/\beta}} = \frac{1-w}{k_1^{1/\beta}} \tag{2.13}$$

$$\tag{2.14}$$

where Q_i is the theoretical quantile for $i = 1,2$ and where k, k_i and w for $i = 1,2$ are specified in (2.8).

The function h_2 and the first derivative of function h_2 have the form

$$h_2 = \beta = \frac{\ln\left[-\ln\left(1-p_1\right)\right] - \ln\left[-\ln\left(1-p_2\right)\right]}{\ln Q_1 - \ln Q_2} \tag{2.15}$$

$$\frac{\mathrm{d}h_2}{\mathrm{d}Q_1} = -\frac{\ln\left[-\ln\left(1-p_1\right)\right] - \ln\left[-\ln\left(1-p_2\right)\right]}{Q_1\left(\ln Q_1 - \ln Q_2\right)^2} = -\frac{\beta^2}{Q_1 k} = -\frac{\beta^2}{\alpha k k_1^{1/\beta}} \tag{2.16}$$

$$\frac{\mathrm{d}h_2}{\mathrm{d}Q_2} = \frac{\ln\left[-\ln\left(1-p_1\right)\right] - \ln\left[-\ln\left(1-p_2\right)\right]}{Q_2\left(\ln Q_1 - \ln Q_2\right)^2} = \frac{\beta^2}{Q_2 k} = \frac{\beta^2}{\alpha k k_2^{1/\beta}}. \tag{2.17}$$

where k, k_i and w for $i = 1, 2$ are specified in (2.8).

From the theorem (2.1) we computed the variances σ_1^2 and σ_2^2 and the covariance $\rho_{12}\sigma_1\sigma_2$.

$$\sigma_1^2 = \frac{\alpha^2 p_1\left[-\ln\left(1-p_1\right)\right]^{2/\beta}}{n\beta^2\left(1-p_1\right)\left[-\ln\left(1-p_1\right)\right]^2} = \frac{\alpha^2 q_1 k_1^{2/\beta}}{n\beta^2 k_1^2} \tag{2.18}$$

$$\sigma_2^2 = \frac{\alpha^2 p_2\left[-\ln\left(1-p_2\right)\right]^{2/\beta}}{n\beta^2\left(1-p_2\right)\left[-\ln\left(1-p_2\right)\right]^2} = \frac{\alpha^2 q_2 k_2^{2/\beta}}{n\beta^2 k_2^2} \tag{2.19}$$

$$\rho_{12}\sigma_1\sigma_2 = \frac{\alpha^2 p_1\left[-\ln\left(1-p_1\right)\right]^{1/\beta}\left[-\ln\left(1-p_2\right)\right]^{1/\beta}}{n\beta^2\left(1-p_1\right)\left[-\ln\left(1-p_1\right)\right]\left[-\ln\left(1-p_2\right)\right]} = \frac{\alpha^2 q_1 k_1^{1/\beta} k_2^{1/\beta}}{n\beta^2 k_1 k_2}. \tag{2.20}$$

We competed the asymptotic covariance matrix of $\hat{\alpha}$ and $\hat{\beta}$ form expression (2.11).

$$\begin{pmatrix} \frac{\mathrm{d}h_1}{\mathrm{d}Q_1} & \frac{\mathrm{d}h_1}{\mathrm{d}Q_2} \\ \frac{\mathrm{d}h_2}{\mathrm{d}Q_1} & \frac{\mathrm{d}h_2}{\mathrm{d}Q_2} \end{pmatrix} \cdot \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \cdot \begin{pmatrix} \frac{\mathrm{d}h_1}{\mathrm{d}Q_1} & \frac{\mathrm{d}h_2}{\mathrm{d}Q_1} \\ \frac{\mathrm{d}h_1}{\mathrm{d}Q_2} & \frac{\mathrm{d}h_2}{\mathrm{d}Q_2} \end{pmatrix} = \begin{pmatrix} \mathrm{Var}\,\hat{\alpha} & \mathrm{Cov}\left(\hat{\alpha},\hat{\beta}\right) \\ \mathrm{Cov}\left(\hat{\alpha},\hat{\beta}\right) & \mathrm{Var}\,\hat{\beta} \end{pmatrix} \tag{2.21}$$

It follows that $\hat{\alpha}$ is asymptotically normal with the mean α and the variance

$$\mathrm{Var}\,\hat{\alpha} = \frac{\alpha^2}{n\beta^2}\left[\frac{wq_1}{k_1}\left(\frac{w}{k_1} + 2\frac{1-w}{k_2}\right) + \frac{\left(1-w\right)^2 q_2}{k_2^2}\right] = \frac{\alpha^2}{n\beta^2}\, g\left(p_1, p_2\right), \tag{2.22}$$

where k, k_i and w for $i = 1, 2$ are specified in (2.8) and

$$q_i = \frac{p_i}{\left(1-p_i\right)} \tag{2.23}$$

for $i = 1, 2$.

It follows that $\hat{\beta}$ is asymptotically normal with the mean β and the variance

$$\mathrm{Var}\,\hat{\beta} = \frac{\beta^2}{nk^2}\left[\frac{q_1}{k_1^2} + \frac{q_2}{k_2^2} - 2\frac{q_1}{k_1 k_2}\right] = \frac{\beta^2}{n}\,\phi\left(p_1, p_2\right), \tag{2.24}$$

where k, k_i, q_i for $i = 1, 2$ are from (2.8) and (2.23)

The quantile estimators $\hat{\alpha}$ and $\hat{\beta}$ are not asymptotically uncorrelated. The asymptotic covariance is

$$\mathrm{Cov}\left(\hat{\alpha},\hat{\beta}\right) = \frac{\alpha}{nk^2}\left[\frac{q_1}{k_1}\left(\frac{k-\ln k_1}{k_2} - \frac{\ln k_1}{k_2} - \frac{k-\ln k_1}{k_1}\right) + \frac{q_2 \ln k_1}{k_2^2}\right], \tag{2.25}$$

where k, k_i and q_i for $i = 1, 2$ are from (2.8) and (2.23).

There we can decide on the optimal choice of p_1 and p_2 considering general variance. Optimal choice of p_1 and p_2 was computed by the minimization of the generalized variance (the determinant of the asymptotic covariance matrix). From the article [4] we obtain the optimal choice of the quantiles as

$$p_1 = 0.23875930, \ p_2 = 0.92656148. \tag{2.26}$$

The values of quantiles which we compute by minimization of the determinant of the asymptotic covariance matrix are

$$p_1 = 0.2624487, \ p_2 = 0.9162927. \tag{2.27}$$

The variances, the covariance and the determinant of the asymptotic covariance matrix have a similar values for this quantiles. The difference between (2.26) and (2.27) was created by numerical error of the R.

2.1.2 Quantiles for the Sum of Probability Equal 1

In the article [11], Pekasiewicz stated a method for choosing two quantile p_1 and p_2, such that $p_1 + p_2 = 1$. We write $p_1 = p$ which gives that $p_2 = 1 - p$. The system of equations has the form of

$$\begin{cases} X_{p,n} = \alpha \left[-\ln(1-p)\right]^{1/\beta} \\ X_{1-p,n} = \alpha \left[-\ln(1-(1-p))\right]^{1/\beta}. \end{cases} \tag{2.28}$$

We derive the estimation of parameter β from (2.4) after substitution $p_1 = p$ and $p_2 = 1 - p$. So we have the estimation of β as

$$\hat{\beta} = \frac{\ln\left[-\ln(1-p)\right] - \ln\left[-\ln p\right]}{\ln X_{p,n} - \ln X_{1-p,n}}. \tag{2.29}$$

The estimation of parameter α we derive from (2.5) after substitution $p_1 = p$ and $p_2 = 1 - p$. So we have the estimation of α as

$$\hat{\alpha} = \exp\left\{\frac{1}{2}\left[\ln X_{p,n} + \ln X_{1-p,n} - \frac{1}{\hat{\beta}}\ln\left[-\ln(1-p)\right] - \frac{1}{\hat{\beta}}\ln\left[-\ln p\right]\right]\right\}. \tag{2.30}$$

The variances and the covariance for this method are similar as for the general case (2.22), (2.24) and (2.25).

2.1.3 Special Quantiles

From the quantile function (1.9) we can deduce the estimate of the parameter α to be independent on the parameter β. The quantile p_2 can be selected so that $\hat{\alpha} = X_{p_2,n}$. It will be when

$$-\ln(1-p_2) = 1.$$

From this implies that $p_2 = (1 - e^{-1}) = 0,632$. Its means that the 0,632th quantile corresponds to the estimation of α.

$$\hat{\alpha} = X_{(1-e^{-1}),n}. \tag{2.31}$$

We derive the estimation of parameter β from (2.4) after substitution $p_2 = (1 - e^{-1})$

$$\hat{\beta} = \frac{\ln(-\ln(1-p_1))}{\ln X_{p_1,n} - \ln X_{(1-e^{-1}),n}} \tag{2.32}$$

for $0 < p_1 < (1 - e^{-1})$.

Let as derive the asymptotic covariance matrix of these estimates. The function h_1 and the first derivative of function h_1 have the form

$$h_1 = \alpha = Q_2 \tag{2.33}$$

$$\frac{\mathrm{d}h_1}{\mathrm{d}Q_1} = 0 \tag{2.34}$$

$$\frac{\mathrm{d}h_1}{\mathrm{d}Q_2} = 1 \tag{2.35}$$

$$\tag{2.36}$$

where Q_i is the theoretical quantile for $i = 1, 2$.

The function h_2 and the first derivative of function h_2 have the form

$$h_2 = \beta = \frac{\ln\left[-\ln\left(1 - p_1\right)\right]}{\ln Q_1 - \ln Q_2} \tag{2.37}$$

$$\frac{\mathrm{d}h_2}{\mathrm{d}Q_1} = -\frac{\ln\left[-\ln\left(1 - p_1\right)\right]}{Q_1\left(\ln Q_1 - \ln Q_2\right)^2} = -\frac{\beta^2}{Q_1 k} = -\frac{\beta^2}{\alpha k k_1^{1/\beta}} \tag{2.38}$$

$$\frac{\mathrm{d}h_2}{\mathrm{d}Q_2} = \frac{\ln\left[-\ln\left(1 - p_1\right)\right]}{Q_2\left(\ln Q_1 - \ln Q_2\right)^2} = \frac{\beta^2}{Q_2 k} = \frac{\beta^2}{\alpha k k_2^{1/\beta}} = \frac{\beta^2}{\alpha k}. \tag{2.39}$$

where k, k_i for $i = 1, 2$ are specified in (2.8).

From equation (2.21) we obtained expression for variances and covariance for the estimation of parameters of special quantiles.

It follows that $\hat{\alpha}$ is asymptotically normal with the mean α and the variance

$$\operatorname{Var}\hat{\alpha} = \frac{\alpha^2 q_2 k_2^{2/\beta}}{n\beta^2 k_2^2} = \frac{\alpha^2 q_2}{n\beta^2}, \tag{2.40}$$

where k_2 and q_2 are specified in (2.8 and (2.23).

It follows that $\hat{\beta}$ is asymptotically normal with the mean β and the variance

$$\operatorname{Var}\hat{\beta} = \frac{\beta^2}{n\left(\ln k_1\right)^2}\left[\frac{q_1}{k_1^2} + \frac{q_2}{k_2^2} - 2\frac{q_1}{k_1 k_2}\right] = \frac{\beta^2}{n\left(\ln k_1\right)^2}\left[\frac{q_1}{k_1^2} + q_2 - 2\frac{q_1}{k_1}\right] = \frac{\beta^2}{n}\phi\left(p_1, p_2\right), \tag{2.41}$$

where k_i and q_i for $i = 1, 2$ are from (2.8) and (2.23).

The percentile estimators $\hat{\alpha}$ and $\hat{\beta}$ are not asymptotically uncorrelated. The asymptotic covariance is

$$\operatorname{Cov}\left(\hat{\alpha}, \hat{\beta}\right) = \frac{\alpha k_2^{1/\beta}}{n k_2 \ln k_1}\left[-\frac{q_1}{k_1} + \frac{q_2}{k_2}\right] = \frac{\alpha}{n \ln k_1}\left[-\frac{q_1}{k_1} + q_2\right], \tag{2.42}$$

where k_i and q_i for $i = 1, 2$ are from (2.8) and (2.23).

The article [14], Seki and Yokoyama proposed $p = 0,31$ for the estimation of $\hat{\beta}$. Which comes from the condition

$$\ln\left[-\ln\left(1 - p\right)\right] = -1.$$

From that we have $p = (1 - e^{-e^{-1}})$.

2.1.4 Method of Quantiles for Intervals

In the article [12], sorting of the sample in two intervals is consider instead of specifying two specific quantiles for p_1 and p_2.

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the ordered random sample, which mean that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$.

We dived the interval of data, which contains the random sample, we divide into m nonoverlapping intervals $I_j = [t_{j-1}, t_j)$, $j = 1, ..., m$. The first value ($X_{(1)}$ or some specification value, for example 0), is entry t_0 and the biggest value t_m is the end of the interval ($X_{(n)}$ or some bigger specific value).

The parameter β on interval can be expressed as

$$\beta = \frac{\ln(-\ln(1 - F(t_{j+1}))) - \ln(-\ln(1 - F(t_j)))}{\ln t_{j+1} - \ln t_j}, \tag{2.43}$$

for $j = 1, ..., p - 1$ the similary as (2.4) and the parameter α on interval

$$\alpha = \frac{t_j}{[-\ln(1 - F(t_j))]^{\frac{1}{\beta}}}, \tag{2.44}$$

for $j = 1, ..., p$ as in (2.7).

In the formulas for estimation of parameters $\hat{\alpha}$ and $\hat{\beta}$ we need to use estimator of the distribution function F. Following [5], we use the modified distribution function $\hat{F}_n^*$ (10).

Modified equations for the estimation of parameters should have the form

$$\hat{\beta} = \frac{\ln\left(\ln\left(1 - \hat{F}_n^*(t_m)\right)^{-1}\right) - \ln\left(\ln\left(1 - \hat{F}_n^*(t_1)\right)^{-1}\right)}{\ln t_m - \ln t_1} \tag{2.45}$$

and

$$\hat{\alpha} = \frac{1}{m} \sum_{j=1}^{m} \frac{t_j}{\left[-\ln\left(1 - \hat{F}_n^*(t_j)\right)\right]^{\frac{1}{\hat{\beta}}}}, \tag{2.46}$$

which is the arithmetic mean of the value of $\hat{\alpha}$ at all m border points.

For the method of quantiles for interval the estimation of parameter α (2.46) is the weighted arithmetic mean. For the general case the estimation of parameter α (2.7) is compute as the geometric mean for 2 percentiles.

It is very important to note that this estimation of parameters α and β is dependent on the choice of the intervals on the choose of intervals. For example, this method is not very good for irregularly distributed data on the interval $[t_0, t_m]$, according to our simulations.

From (1.9) we should derive or if $m = 1$ in the formula (2.46)

$$\hat{\alpha}^* = \frac{\hat{F}_n^*(t_1)}{[-\ln(1 - t_1)]^{\frac{1}{\hat{\beta}}}}. \tag{2.47}$$

This means that the estimation of parameter α can be computed for one boundary point (t_1) or as the arithmetic mean of all m boundary points.

2.2 Method of Maximum Likelihood

The primary sources of this section were the books [2], [7] and [13].

The maximum likelihood (ML) is one the most popular method for estimation of the unknown parameters of the distribution.
Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample from the distribution with the density function $f(x, \boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$ is from the parameter space Ω.

Definition 2.3. Let $X_1, \ldots, X_n$ be a random sample from the distribution with the density function $f(x, \boldsymbol{\theta})$ with respect to a σ-finite measure μ. Suppose the following conditions hold.

a) The parameter space Ω is an open and non empty in $\mathbb{R}^n$ and $\boldsymbol{\theta} \in \Omega$.

b) The set $M = \{x : f(x, \boldsymbol{\theta}) > 0\}$ is independent of $\boldsymbol{\theta}$.

c) For almost every $x \in M$ with respect to μ and there exist the partial derivations $f_i'(x, \boldsymbol{\theta}) = \frac{\partial f(x, \boldsymbol{\theta})}{\partial \theta_i}$ for $i = 1, \ldots, m$.

d) For every $\boldsymbol{\theta} \in \Omega$ hold $\int_M f_i'(x, \boldsymbol{\theta}) \, \mathrm{d}\mu(x) = 0$ for $i = 1, \ldots, m$.

e) For every pair (i, j) exists the finite integral

$$I_{ij}(\boldsymbol{\theta}) = \int_M \frac{f_i'(\mathbf{X}, \boldsymbol{\theta}) \, f_j'(\mathbf{X}, \boldsymbol{\theta})}{f^2(\mathbf{X}, \boldsymbol{\theta})} f(\mathbf{X}, \boldsymbol{\theta}) \, \mathrm{d}\mu(\mathbf{X})$$

for $i, j = 1, \ldots, m$, where $f(\mathbf{X}, \boldsymbol{\theta}) = f(X_1, \boldsymbol{\theta}) \cdot \ldots \cdot f(X_n, \boldsymbol{\theta})$.

f) The matrix $\mathbf{I}_n(\boldsymbol{\theta}) = [I_{ij}(\boldsymbol{\theta})]_{i,j=1}^m$ is positive definite for every $\boldsymbol{\theta} \in \Omega$.

Then the system $\{f(x, \boldsymbol{\theta}), \boldsymbol{\theta} \in \Omega\}$ is called *regular* and $\mathbf{I}_n(\boldsymbol{\theta})$ is called the *Fisher information matrix.*

Now we present the assumptions that we will continue to assume in the following section:

(A0) Let $\theta_1, \theta_2 \in \Omega$. Then $f(x, \theta_1) = f(x, \theta_2)$ $[\mu]$ if and only if $\theta_1 = \theta_2$.

(A1) The set $M = \{x : f(x, \boldsymbol{\theta}) > 0\}$ is independent of $\boldsymbol{\theta}$.

(A2) $X_1, \ldots, X_n$, where X_i are iid, have the distribution with the density function $f(x, \boldsymbol{\theta})$ with respect to a σ-finite measure μ.

(A3) The parameter space Ω contains an open set ω of which the true parameter value $\boldsymbol{\theta}_0$ is an interior point.

We use the term *likelihood* for the density function dependent on the distributional parameters $\theta_1, \ldots, \theta_m$. The *likelihood element* L_i is the likelihood of an individual observation. We have n independent observation (length of $\mathbf{X}$) so likelihood is a product of all L_i.

For observation x_i of the random variable X_i the likelihood element is defined as

$$L_i(\boldsymbol{\theta}) := f(x_i, \boldsymbol{\theta}), \tag{2.48}$$

where $f(x, \boldsymbol{\theta})$ is density function.

The *likelihood function* is

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} L_i(\boldsymbol{\theta}). \tag{2.49}$$

Often the *log-likelihood function* is used since it is better for computing, defined as

$$\mathcal{L}(\boldsymbol{\theta}) := \ln L(\boldsymbol{\theta}) = \sum_{i=1}^{n} \ln L_i(\boldsymbol{\theta}). \tag{2.50}$$

The maximum likelihood estimation (MLE) is the value of parameter $\boldsymbol{\theta}$ such that the log-likelihood function $\mathcal{L}(\boldsymbol{\theta})$ has the maximum value in the parametric space Ω. Extreme values of the log-likelihood function $\mathcal{L}(\boldsymbol{\theta})$ are specified using the stationary points.

The extremal points (maximum) of $L(\boldsymbol{\theta})$ and $\mathcal{L}(\boldsymbol{\theta})$ will be the same, because the logarithmic transformation is isotonic.

A *system of likelihood equations* is obtained by taking a partial derivative of $\mathcal{L}(x, \boldsymbol{\theta})$ with respect to parameters $\theta_1, \ldots, \theta_m$

$$\frac{\partial \mathcal{L}(\boldsymbol{\theta})}{\partial \theta_i} = 0, \tag{2.51}$$

for $i = 1, \ldots, m$.

For the two-parameter Weibull model we obtain stationary points as solutions of the system of likelihood equations

$$\frac{\partial \mathcal{L}(x, \alpha, \beta)}{\partial \alpha} = 0 \text{ and } \frac{\partial \mathcal{L}(x, \alpha, \beta)}{\partial \beta} = 0.$$

Theorem 2.4. *Let the system $\{f(x, \boldsymbol{\theta}), \boldsymbol{\theta} \in \Omega\}$ is regular and has the Fisher information matrix $\mathbf{I}_n(\boldsymbol{\theta})$. Let (A0) - (A3) be satisfied. Suppose that the following conditions hold.*

a) For almost every $x \in M$ exists the third derivative $\frac{\partial^3 f(x,\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j \partial \theta_k}$, where $i, j, k = 1, 2$, for every $\boldsymbol{\theta} \in \omega$.

b) For every $\boldsymbol{\theta} \in \omega$ hold

$$\int_M f''_{ij}(x, \boldsymbol{\theta}) \, \mathrm{d}\mu(x) = 0, \; i, j = 1, \ldots, m.$$

c) For every $i, j, k = 1, \ldots, m$ exists the function $M_{ijk}(x) \geq 0$ such that

$$E_{\boldsymbol{\theta}_0} M_{ijk}(x) < \infty$$

and

$$\left| \frac{\partial^3 f(x, \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j \partial \theta_k} \right| \leq M_{ijk}(x),$$

for every $\boldsymbol{\theta} \in \omega$ and almost every $x \in M$.

Then the following hold.

i) If $n \to \infty$, then for every $\epsilon > 0$ there exists with probability tending to 1 the solution of likelihood equations $\hat{\boldsymbol{\theta}}_n$ such that $\| \hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 \| < \epsilon$.

ii) Exists for any sufficiently large n and for each value of $\mathbf{X}$ such $\hat{\boldsymbol{\theta}}_n$ root of system of likelihood equations that $\hat{\boldsymbol{\theta}}_n$ is consistent estimates of parameter $\boldsymbol{\theta}_0$, then

$$\sqrt{n}\left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \xrightarrow{d} \mathrm{N}\left(0, \mathbf{I}^{-1}(\boldsymbol{\theta}_0)\right).$$

Proof. Omitted. The proofs can be found in [2] or [7]. □

Regularity of Weibull Distribution

We need to show that the two-parameter Weibull distribution and parameter $\boldsymbol{\theta} = (\alpha, \beta)$ from the parameter space Ω is regular system. We must show that the conditions from Definition (2.3) hold:

a) Two parameters are defined on (1.2) as $\alpha > 0$ and $\beta > 0$. It means that $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ and exactly $\Omega = (0; \infty) \times (0; \infty)$. So the parameter space Ω is an open and non empty subset of $\mathbb{R}^2$.

b) The set $M = \{x : f(x, \boldsymbol{\theta}) > 0\}$ is independent of $\boldsymbol{\theta}$ because the density function for $f(x, \alpha, \beta)$ is positive for all $x \geq 0$ independently on $\boldsymbol{\theta}$.

c) The function is continuous relative to the parameters for every $x \in M$ with respect parameters α and β to μ. The partial derivatives of the density function $f_i'(x, \boldsymbol{\theta})$ are given in (2.53) and (2.54) for $i = 1, 2$.

d) For every $\boldsymbol{\theta} \in \Omega$ it holds that

$$\int_M f_i'(x, \boldsymbol{\theta}) \, \mathrm{d}\mu(x) = 0$$

for $i = 1, 2$ and we can see it in (2.53) and (2.54).

e) For every pair (i, j) the finite integral $I_{ij}(\boldsymbol{\theta})$ for $i, j = 1, 2$ exists, as we can see from (2.58).

f) We can check that the matrix $\mathbf{I}_n(\alpha, \beta)$ is positive definite for every α and β from Ω by Sylvester's criterion. This is a necessary and sufficient condition to determine whether the matrix is positive definite.

$$\det D_1 = \det\left(-E\left[\frac{\partial^2 \mathcal{L}(\alpha, \beta)}{\partial \alpha^2}\right]\right) = \frac{n\beta^2}{\alpha^2} > 0,$$

where we use element $(1, 1)$ of the Fisher information matrix (2.59) and

$$\begin{aligned}
\det D_2 &= \det \mathbf{I}_n(\alpha, \beta) = \det \begin{pmatrix} \frac{n\beta^2}{\alpha^2} & -\frac{n}{\alpha}\Gamma'(2) \\ -\frac{n}{\alpha}\Gamma'(2) & \frac{n}{\beta^2}\left(\Gamma''(2) + 1\right) \end{pmatrix} \\
&= \frac{n^2}{\alpha^2}\left(\Gamma''(2) - \left(\Gamma'(2)\right)^2 + 1\right) \\
&= \frac{n^2}{\alpha^2}\left(-2\gamma + \gamma^2 + \frac{\pi^2}{6} - (1-\gamma)^2 + 1\right) = \frac{n^2}{\alpha^2}\frac{\pi^2}{6} > 0,
\end{aligned}$$

where $\gamma \doteq 0,5772$ is *Euler constant* and the Fisher information matrix is from (2.59). Both determinants are positive for every $\boldsymbol{\theta} \in \Omega$ and $n > 0$. It is mean that $\mathbf{I}_n(\alpha, \beta)$ two-parameter Weibull distribution is positive definite for every α and β from Ω and $n > 0$.

Therefore we can conclude that the two-parameter Weibull distribution is regular and MLE estimate are asymptotically normal and Theorem (2.4) holds.

2.2.1 MLE for the Two-Parameter Weibull Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from the two-parameter Weibull distribution.

We need to find the estimation of α and β such that we maximize (2.50). The solution of (2.2) $\hat{\alpha}$ and $\hat{\beta}$ are stationary points of log-likelihood function and for the maximum likelihood estimation (MLE) might include the global maximum.

The log-likelihood function for the standard Weibull model has the form of

$$\begin{aligned}
\mathcal{L}(\alpha, \beta) &= \sum_{i=1}^{n} \ln \left[\frac{\beta x_i^{\beta-1}}{\alpha^\beta} \exp \left[-\left(\frac{x_i}{\alpha} \right)^\beta \right] \right] \\
&= \sum_{i=1}^{n} \left[\ln \beta - \beta \ln \alpha + (\beta - 1) \ln x_i - \left(\frac{x_i}{\alpha} \right)^\beta \right] \\
&= n \left[\ln \beta - \beta \ln \alpha \right] + (\beta - 1) \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\alpha} \right)^\beta . \qquad (2.52)
\end{aligned}$$

Therefore the system of likelihood equations is

$$\frac{\partial \mathcal{L}(\alpha, \beta)}{\partial \alpha} = -\frac{n\beta}{\alpha} + \frac{\beta}{\alpha} \sum_{i=1}^{n} \left(\frac{x_i}{\alpha} \right)^\beta = 0, \qquad (2.53)$$

$$\frac{\partial \mathcal{L}(\alpha, \beta)}{\partial \beta} = \frac{n}{\beta} - n \ln \alpha + \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\alpha} \right)^\beta \ln \left(\frac{x_i}{\alpha} \right) = 0. \qquad (2.54)$$

The Hessian matrix is used for finding the solution of the system of likelihood equations. The *Hessian matrix* of $\mathcal{L}(x, \alpha, \beta)$ has the form of

$$\mathbf{H}(x, \boldsymbol{\theta}) = \frac{\partial^2 \mathcal{L}(x, \boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}, \qquad (2.55)$$

where $\boldsymbol{\theta} = (\alpha, \beta)$, with elements

$$\mathbf{H}_{11} = \frac{\partial^2 \mathcal{L}(x, \boldsymbol{\theta})}{\partial \alpha^2}, \; \mathbf{H}_{12} = \mathbf{H}_{21} = \frac{\partial^2 \mathcal{L}(x, \boldsymbol{\theta})}{\partial \alpha \, \partial \beta}, \; \mathbf{H}_{22} = \frac{\partial^2 \mathcal{L}(x, \boldsymbol{\theta})}{\partial \beta^2}. \qquad (2.56)$$

For the standard model the Hessian matrix consists of these elements

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \alpha^2} &= \frac{\beta}{\alpha^2} \left[n - (\beta + 1) \sum_{i=1}^{n} \left(\frac{x_i}{\alpha} \right)^\beta \right], \\
\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \alpha \partial \beta} = \frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \beta \partial \alpha} &= \frac{1}{\alpha} \left[-n + \sum_{i=1}^{n} \left(\frac{x_i}{\alpha} \right)^\beta + \beta \sum_{i=1}^{n} \left(\frac{x_i}{\alpha} \right)^\beta \ln \left(\frac{x_i}{\alpha} \right) \right], \\
\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \beta^2} &= -\frac{n}{\beta^2} - \sum_{i=1}^{n} \left(\frac{x_i}{\alpha} \right)^\beta \left[\ln \left(\frac{x_i}{\alpha} \right) \right]^2 .
\end{aligned}$$

We can derive the Fisher information matrix from the Hessian matrix [2]. For the Fisher information matrix of $\mathcal{L}(\theta)$ it is true that

$$\mathbf{I}_n(\boldsymbol{\theta}) = -E\left[\mathbf{H}(\boldsymbol{\theta})\right]. \qquad (2.57)$$

For the two-parameters Weibull model the elements of $\mathbf{I}_n(\boldsymbol{\theta})$ have the form

$$
\begin{aligned}
-E\left[\frac{\partial^2\mathcal{L}(\boldsymbol{\theta})}{\partial\alpha^2}\right] =& \frac{-n\beta}{\alpha^2}+\frac{\beta(\beta+1)}{\alpha^2}\sum_{i=1}^n E\left[\left(\frac{X_i}{\alpha}\right)^\beta\right] = \frac{-n\beta}{\alpha^2}+\frac{\beta(\beta+1)}{\alpha^2}\sum_{i=1}^n\left(\frac{1}{\alpha}\right)^\beta E\left[X_i^\beta\right] \\
=& \frac{-n\beta}{\alpha^2}+\frac{\beta(\beta+1)}{\alpha^2}\sum_{i=1}^n\left(\frac{1}{\alpha}\right)^\beta\alpha^\beta\Gamma\left(1+\frac{\beta}{\beta}\right) = \frac{-n\beta}{\alpha^2}+\frac{n\beta(\beta+1)}{\alpha^2} = \frac{n\beta^2}{\alpha^2} \\
-E\left[\frac{\partial^2\mathcal{L}(\boldsymbol{\theta})}{\partial\alpha\partial\beta}\right] =& -E\left[\frac{\partial^2\mathcal{L}(\boldsymbol{\theta})}{\partial\beta\partial\alpha}\right] = \frac{n}{\alpha}-\frac{1}{\alpha}\sum_{i=1}^n E\left[\left(\frac{X_i}{\alpha}\right)^\beta\right]-\frac{\beta}{\alpha}\sum_{i=1}^n E\left[\left(\frac{X_i}{\alpha}\right)^\beta\ln\left(\frac{X_i}{\alpha}\right)\right] \\
=& \frac{n}{\alpha}-\frac{1}{\alpha}\sum_{i=1}^n\left(\frac{1}{\alpha}\right)^\beta\alpha^\beta\Gamma\left(1+\frac{\beta}{\beta}\right)-\frac{\beta}{\alpha}\sum_{i=1}^n\left(\frac{1}{\alpha}\right)^\beta E\left[X_i^\beta\ln X_i\right] \\
&+\frac{\beta}{\alpha}\sum_{i=1}^n\left(\frac{1}{\alpha}\right)^\beta\ln\alpha E\left[X_i^\beta\right] = -\frac{n}{\alpha}\Gamma'(2) \\
-E\left[\frac{\partial^2\mathcal{L}(\boldsymbol{\theta})}{\partial\beta^2}\right] =& \frac{n}{\beta^2}+\sum_{i=1}^n\left(\frac{1}{\alpha}\right)^\beta E\left[X_i^\beta\left(\ln\left(\frac{X_i}{\alpha}\right)\right)^2\right] = \frac{n}{\beta^2}+\frac{n}{\alpha^\beta}E\left[X^\beta(\ln X)^2\right] \\
&-2\frac{n}{\alpha^\beta}\ln\alpha E\left[X^\beta\ln X\right]+\frac{n}{\alpha^\beta}(\ln\alpha)^2E\left[X^\beta\right] = \frac{n}{\beta^2}\left(\Gamma''(2)+1\right). \quad (2.58)
\end{aligned}
$$

therefore

$$\mathbf{I}_n(\boldsymbol{\theta}) = \begin{pmatrix} \frac{n\beta^2}{\alpha^2} & -\frac{n}{\alpha}\Gamma'(2) \\ -\frac{n}{\alpha}\Gamma'(2) & \frac{n}{\beta^2}\left(\Gamma''(2)+1\right) \end{pmatrix} \quad (2.59)$$

This implies that the asymptotic covariance matrix of the estimated α and β is

$$\mathbf{I}_n^{-1}(\boldsymbol{\theta}) = \frac{1}{\frac{n^2}{\alpha^2}\frac{\pi^2}{6}}\begin{pmatrix} \frac{n}{\beta^2}\left(\Gamma''(2)+1\right) & \frac{n}{\alpha}\Gamma'(2) \\ \frac{n}{\alpha}\Gamma'(2) & \frac{n\beta^2}{\alpha^2} \end{pmatrix} \quad (2.60)$$

with determinant

$$\det\mathbf{I}_n^{-1}(\boldsymbol{\theta}) = \left(\frac{6\alpha^2}{n^2\pi^2}\right)^2\left(\frac{n^2\beta^2}{\alpha^2\beta^2}\left(\Gamma''(2)+1\right)-\frac{n^2}{\alpha^2}\left(\Gamma'(2)\right)^2\right) = \frac{\alpha^2}{n^2}\frac{6}{\pi^2} \quad (2.61)$$

If we compare the determinant (2.61) with the values $\frac{n^2}{\alpha^2}$ from the Table 5.3 we can see that the MLE for two-parameter Weibull model is a better method, because this method has smallest determinant of the asymptotic covariance matrix.

2.2.2 Iterated MLE

The system of equations (2.53) and (2.54) is not linear, this means that for finding solution we must use the iterative methods.

From (2.53) for MLE of α the following holds

$$\alpha = \left[\frac{1}{n}\sum_{i=1}^n x_i^\beta\right]^{\frac{1}{\beta}}. \quad (2.62)$$

After substitution (2.62) into (2.54), we have the equation

$$\frac{1}{\beta}+\frac{1}{n}\sum_{i=1}^n\ln x_i-\frac{\sum_{i=1}^n x_i^\beta\ln x_i}{\sum_{i=1}^n x_i^\beta} = 0. \quad (2.63)$$

The solution of this equation has to be found iteratively in the reduced parametric space $\Omega^* \subset \Omega$.

The solution is equivalent to the estimation of parameter β by the maximum likelihood method. The estimation of α is obtained from (2.62) after substituting $\hat{\beta}$ for β.

Another option is to maximize the equation

$$\mathcal{L}(\beta) = n\left(\ln\beta - \ln\left[\frac{1}{n}\sum_{i=1}^{n} x_i^{\beta}\right]\right) + (\beta - 1)\sum_{i=1}^{n}\ln x_i - n, \tag{2.64}$$

which is the log-likelihood function (2.52) after substitution (2.62) and its gradient (2.63).

The initial value for iteration we can be the estimation from method of quantiles.

2.3 Weibull Probability Plot

The graphical method of estimation of parameters can be besed on the *Weibull probability plot* [4].

By logarithming both sides of the quantile function (1.9) we get a linear relation between $\ln Q(p,\alpha,\beta)$ and $\ln[-\ln(1-p)]$

$$\beta\ln Q(p,\alpha,\beta) - \beta\ln\alpha = \ln[-\ln(1-p)], \tag{2.65}$$

which allows us to estimate the parameters α and β.

For the Weibull probability plot the observed variable is z and the response variable y.

$$\begin{aligned} z &= \ln Q(p,\alpha,\beta) \\ y &= \ln[-\ln(1-p)]. \end{aligned}$$

In the plot we use the sample quantile of our data for the regular spacing of probability $p \in (0,1)$. When the random selection of the two-parameters Weibull model is assumed, the transformation will be assumed to have a linear dependence as in the Figure 2.1.

2.3.1 Estimation by Method of Least Squares

We take the parameters from approximation of (2.65) and we use the simple linear regression (SLR) to estimate the parameters.

Denoted $\mathbf{X}$ and $\mathbf{Y}$

$$\mathbf{X} = \begin{pmatrix} 1 & \ln\hat{Q}(p_1,\alpha,\beta) \\ 1 & \ln\hat{Q}(p_2,\alpha,\beta) \\ \vdots & \vdots \\ 1 & \ln\hat{Q}(p_n,\alpha,\beta) \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} \ln[-\ln(1-p_1)] \\ \ln[-\ln(1-p_2)] \\ \vdots \\ \ln[-\ln(1-p_n)] \end{pmatrix}.$$

Then

$$\mathbf{Y} = \mathbf{X}\boldsymbol{c} + \mathbf{e}$$

is called the *linear regression model* (LRM), where $\boldsymbol{c} = (a,b)^T$ is the vector of unknown parameters and $\mathbf{e}$ is the random vector satisfying $E[\mathbf{e}] = 0$ and $\operatorname{Var}\mathbf{e} = \sigma^2\mathbf{I}$, where $\sigma^2 > 0$ is the unknown parameter too and $n \geq 3$.

By comparing (2.65) we obtain

$$\alpha = \exp\left(-\frac{a}{b}\right), \ \beta = b. \tag{2.66}$$

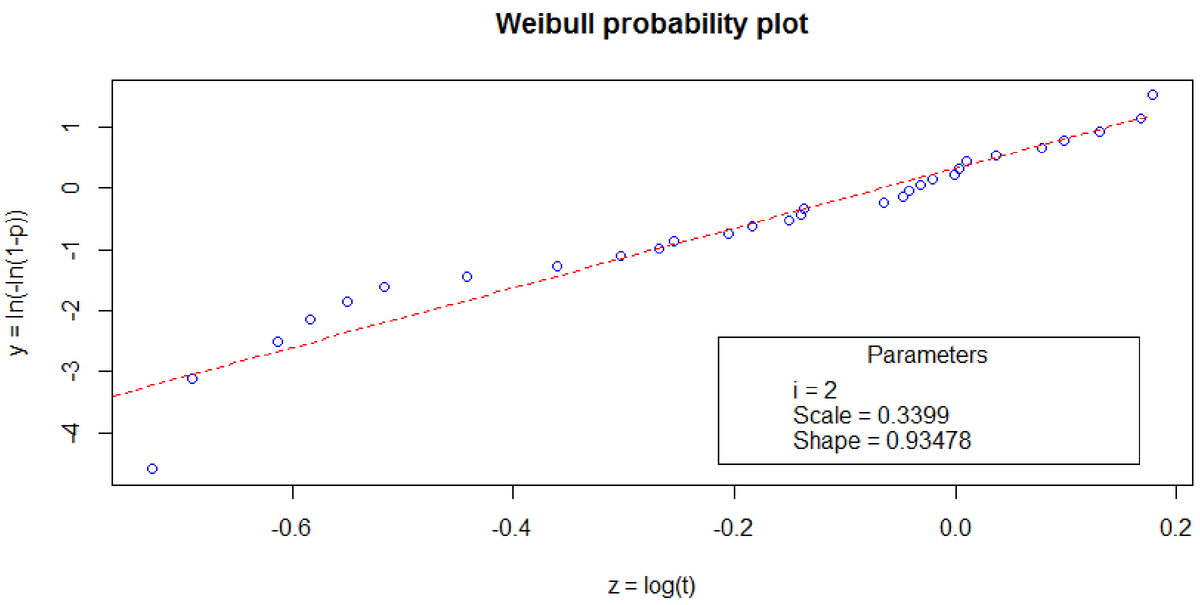


Figure 2.1: The Weibull probability plot for random sample $\mathbf{X}_2$ of size 30 from the two-parametric Weibull distribution with $\alpha = 1$ and $\beta = 5$. In the box the index of sample in dataSIM2.csv and estimates of α and β by least square method are given.

From the method of least squares the estimation of the parameters a and b can be expressed as

$$\hat{\mathbf{c}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'\mathbf{Y},$$

where $\hat{\mathbf{c}} = (\hat{a}, \hat{b})'$.

$$\hat{b} = \hat{\beta} = \frac{n \sum_{i=1}^{n} \ln \hat{Q}\left(p_i\right) \ln\left[-\ln\left(1-p_i\right)\right] - \sum_{i=1}^{n} \ln \hat{Q}\left(p_i\right) \sum_{i=1}^{n} \ln\left[-\ln\left(1-p_i\right)\right]}{n \sum_{i=1}^{n} \left(\ln \hat{Q}\left(p_i\right)\right)^2 - \left(\sum_{i=1}^{n} \ln \hat{Q}\left(p_i\right)\right)^2} \tag{2.67}$$

$$\hat{a} = \frac{\sum_{i=1}^{n} \left(\ln \hat{Q}\left(p_i\right)\right)^2 \sum_{i=1}^{n} \ln\left[-\ln\left(1-p_i\right)\right] - \sum_{i=1}^{n} \ln \hat{Q}\left(p_i\right) \sum_{i=1}^{n} \ln \hat{Q}\left(p_i\right) \ln\left[-\ln\left(1-p_i\right)\right]}{n \sum_{i=1}^{n} \left(\ln \hat{Q}\left(p_i\right)\right)^2 - \left(\sum_{i=1}^{n} \ln \hat{Q}\left(p_i\right)\right)^2} \tag{2.68}$$

We can compute the estimation of the parameter a alternatively as

$$\hat{a} = \frac{1}{n} \sum_{i=1}^{n} \ln\left[-\ln\left(1-p_i\right)\right] - \hat{b}\frac{1}{n} \sum_{i=1}^{n} \ln \hat{Q}\left(p_i\right) \tag{2.69}$$

and we use this form of $\hat{a}$ and the first equation from (2.66) to derive the parameter α as

$$\hat{\alpha} = \exp\left[-\ln \prod_{i=1}^{n} \left[-\ln\left(1-p_i\right)\right]^{\frac{1}{\hat{\beta}n}} + \ln \prod_{i=1}^{n} \left(\hat{Q}\left(p_i\right)\right)^{\frac{1}{n}}\right] = \prod_{i=1}^{n} \left(\frac{\hat{Q}\left(p_i\right)}{\left[-\ln\left(1-p_i\right)\right]^{\frac{1}{\hat{\beta}}}}\right)^{\frac{1}{n}}, \tag{2.70}$$

which is the geometric mean of the estimation of parameter α for n quantiles. Unlike (2.7), the previous estimate of the parameter α is not computed from just 2 quantiles. The difference between (2.7) and (2.70) is that (2.7) is the weighted arithmetic mean and a (2.70) is the usual geometric mean.

Chapter 3

Goodness of Fit Testing

The primary source of this chapter was the book [13].

Let $X_1, \ldots, X_n$ be a random sample from the two-parameter Weibull models such that the random samples comes from the Weibull distribution, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$ is from the parameter space Ω.

Before applying the Weibull distribution, it is very important to test the hypothesis that the real data have the Weibull distribution. We can used the goodness of fit test for testing the null hypothesis. We have a several ways to test the hypothesis, for example, the test of χ^2-type which is independent of known or unknown parameters of the model of the null hypothesis. If we known all parameters we will be use test procedure for fully specified distribution. If we do not know some or all the parameters of the model we will use the procedure for a composite hypothesis. If we known all parameters we will use test procedure for fully specified distribution. If we do not known some of parameters we will use test procedure for composite specified distribution.

3.1 Test of χ^2-type

The χ^2 goodness of fit test requires large sample size [13]. The test can be used when the hypothetical distribution is fully specified or a composite hypothesis. The null hypothesis is

$$H_0 : F_X(x) = F_0(x; \boldsymbol{\theta}).$$

We group sampled data from distribution with the distribution function $F_X(x)$ to k classes, where $k \geq 2$. n_i is the empirical frequency in class $(i = 1, \ldots, k)$ and x_{i-1} and x_i are its limit values. It is required that $n_i \geq 10$ of all i because the test only holds asymptotically or else we have to remake the classes. We compute the expected frequency for each class as

$$E_i = n\left(F_0(x_i; \boldsymbol{\theta}) - F_0(x_{i-1}; \boldsymbol{\theta})\right)$$

or

$$E_i = n\left(F_0\left(x_i; \hat{\boldsymbol{\theta}}\right) - F_0\left(x_{i-1}; \hat{\boldsymbol{\theta}}\right)\right)$$

demands E_i not less than 1 and not more then 20 % of the E_i which has to be less than 5 or else we have to remake classes.

The test statistic is

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - E_i)^2}{E_i}, \tag{3.1}$$

We rejecting the null hypothesis $H_0 : F_X(x) = F_0(x;\boldsymbol{\theta})$ with level α when

$$\chi^2 > \chi^2_{k-m-1,1-\alpha},$$

where m is the number of unknown parameters in $\boldsymbol{\theta}$.

χ^2-tests are generally less powerful than EDF tests and special purpose tests of fit [13].

3.2 EDF Statistic

A statistic measuring the difference between $\hat{F}_n(x)$ and a hypothetical distribution (the cumulative distribution function (CDF)) $F_0(x;\boldsymbol{\theta})$ will be called an *EDF statistic*.

3.2.1 Kolmogorov-Smirnov Statistic

Denote by D^+ the largest vertical difference between $\hat{F}_n(x)$ and $F_0(x;\boldsymbol{\theta})$ when $\hat{F}_n(x)$ is greater than $F_0(x;\boldsymbol{\theta})$:

$$D^+ := \sup_x \{\hat{F}_n(x) - F_0(x;\boldsymbol{\theta})\}. \tag{3.2}$$

Similarly, denote the largest vertical difference between $\hat{F}_n(x)$ and $F_0(x;\boldsymbol{\theta})$ when $\hat{F}_n(x)$ is less then $F_0(x;\boldsymbol{\theta})$:

$$D^- := \sup_x \{F_0(x;\boldsymbol{\theta}) - \hat{F}_n(x)\}. \tag{3.3}$$

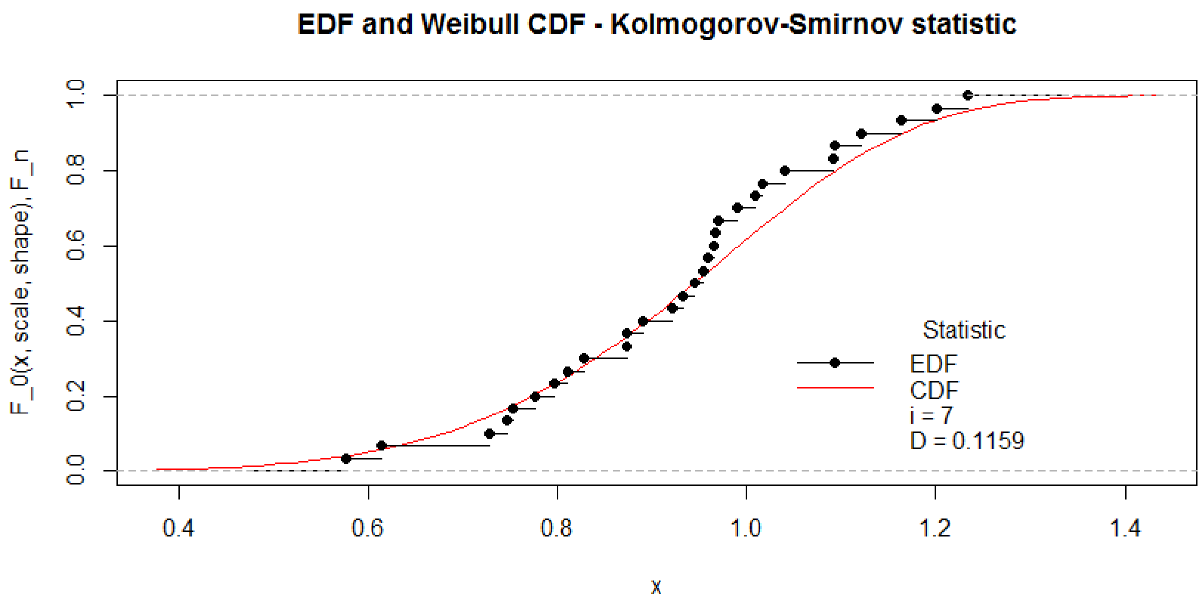


Figure 3.1: The graph of EDF for sample $\mathbf{X}_7$ of size 30 from two-parameter Weibull distribution with $\alpha = 1$ and $\beta = 5$. In the legend the index of sample in dataSIM2.csv and the value of test statistic D are given.

The *Kolmogorov-Smirnov statistic* is defined as

$$D := \sup_x \{|\hat{F}_n(x) - F_0(x;\boldsymbol{\theta})|\} = \max\{D^+, D^-\}. \tag{3.4}$$

If $F_n(x)$ is a continuous function then it is possible to compute the Kolmogorov-Smirnov statistic using formulas. First set

$$Z_i = F_0(X_i; \boldsymbol{\theta}), \tag{3.5}$$

then

$$D^+ = \max_{1 \le i \le n} \{\frac{i}{n} - Z_i\}, \tag{3.6}$$

$$D^+ = \max_{1 \le i \le n} \{Z_i - \frac{i}{n}\}, \tag{3.7}$$

$$D = \max\{D^+, D^-\}. \tag{3.8}$$

3.2.2 Anderson-Darling Statistic

The *Anderson-Darling statistic* is defined as

$$A^2 := n \int_{-\infty}^{\infty} \frac{[F_n(x) - F_0(x;\boldsymbol{\theta})]^2}{F_0(x;\boldsymbol{\theta})[1 - F_0(x;\boldsymbol{\theta})]} dF_0(x;\boldsymbol{\theta}). \tag{3.9}$$

If we use Theorem 1.1 from [2], which says that the integral and derivative (with respect to $\boldsymbol{\theta}$) can be interchanged, we can rewrite the equation (3.9) as

$$\begin{aligned}
A^2 &:= n \int_0^{\infty} \frac{[F_n(x) - F_0(x;\boldsymbol{\theta})]^2}{F_0(x;\boldsymbol{\theta})[1 - F_0(x,\boldsymbol{\theta})]} f_0(x)\, dx = \\
&= n \int_0^{\infty} \frac{\left[F_n(x) - \left(1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)\right]^2}{\left(1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)\left[1 - \left(1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)\right]} \frac{\beta x^{\beta-1}}{\alpha^{\beta}} \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right] dx = \\
&= n\Bigg[\int_0^{X_{(1),n}} \frac{\left(-1 + \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)^2}{\left(1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)} \frac{\beta x^{\beta-1}}{\alpha^{\beta}} dx + \\
&\quad + \sum_{i=1}^{n-1} \int_{X_{(i),n}}^{X_{(i+1),n}} \frac{\left(\frac{i}{n} - 1 + \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)^2}{\left(1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)} \frac{\beta x^{\beta-1}}{\alpha^{\beta}} dx + \\
&\quad + \int_{X_{(n),n}}^{\infty} \frac{\left(\exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)^2}{\left(1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)} \frac{\beta x^{\beta-1}}{\alpha^{\beta}} dx\Bigg] = \\
&= n\Bigg[\int_0^{X_{(1),n}} \left(1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right) \frac{\beta x^{\beta-1}}{\alpha^{\beta}} dx + \\
&\quad + \sum_{i=1}^{n-1} \int_{X_{(i),n}}^{X_{(i+1),n}} \frac{\left(\frac{i}{n} - 1 + \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)^2}{\left(1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)} \frac{\beta x^{\beta-1}}{\alpha^{\beta}} dx + \\
&\quad + \int_{X_{(n),n}}^{\infty} \frac{\left(\exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)^2}{\left(1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right)} \frac{\beta x^{\beta-1}}{\alpha^{\beta}} dx\Bigg]
\end{aligned} \tag{3.10}$$

If $F_n(x)$ is a continuous function then the following formula is possible:

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}\{(2i-1)\ln Z_i + (2n+1-2i)\ln(1-Z_i)\}, \tag{3.11}$$

where Z_i is specified in (3.5).

3.2.3 Test of Fully Specified Hypothesis Based on EDF Statistic

Here we consider the null hypothesis

$$H_0 : F(x) = F_0(x; \boldsymbol{\theta}_0),$$

where $\boldsymbol{\theta}_0$ is fully specified. Then we use the following algorithm to compute tests:

1. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the ordered random sample.
2. Calculate $Z_i = F_0\left(X_{(i)}; \boldsymbol{\theta}_0\right)$ for $i = 1, 2, \ldots, n$.
3. Calculate the statistics (3.4) and (3.10) or for continuous function $F_n(x)$ (3.8) and (3.11).
4. Modify the test statistic as in Table 3.1 using the modification for percentage points. If the statistic exceeds the value in the upper tail given at significance level α then H_0 is rejected at significance level α.

Table 3.1: Modification and percentage points of EDF statistic for testing a hypothesis on a completely specified distribution [13]

Statistic		Significance level α			
T	Modified form T*	0,10	0,05	0,025	0,01
		Upper tail percentage points			
D	$\mathrm{D}\,(\sqrt{n} + 0,12 + 0,11/\sqrt{n})$	1,224	1,358	1,480	1,628
A^2	for all $n \geq 5$	1,933	2,492	3,070	3,880
		Lower tail percentage points			
D	$\mathrm{D}\,(\sqrt{n} + 0,275 - 0,04/\sqrt{n})$	0,571	0,520	0,481	0,441
A^2	for all $n \geq 5$	0,346	0,283	0,240	0,201

3.2.4 Tests of Composite Hypothesis Based on EDF Statistic

The composite hypothesis mean that some or all the parameters θ_i are unknown and we must estimate them first and then compute $F_0\left(X_{(i)}; \boldsymbol{\theta}\right)$. The null hypothesis in this section has the form
$H_0 : X_1, X_2, \ldots, X_n$ for random sample from two-parameter Weibull distribution with parameters α and β.

Here we follow the results from [13] for the test of null hypothesis for which the Table 3.2 for the type-I extreme value distribution of the maximum can be used. Let

$$Y = -\ln X,$$

where X has the two-parameter Weibull distribution with parameters α and β. The CDF of Y has form

$$F(x;\eta,\phi) = \exp\left\{ -\exp\left[-\frac{y-\eta}{\phi}\right]\right\}$$

for $y \in \mathbb{R}$ with $\eta = -\ln\alpha$ and $\phi = 1/\beta$.

A test for hypothesis H_0 is made by testing that Y has the above extreme-value distribution, with both parameters unknown. We used the following algorithm to compute the tests:

1. Make the transformation $Y_i = -\ln X_i$ for $i = 1, 2, \ldots, n$.

2. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be ordered.

3. Estimate the unknown parameters

$$\hat{\phi} = \frac{1}{n}\sum_{j=1}^{n} Y_j - \frac{\sum_{j=1}^{n} Y_j \exp\left(-Y_j/\hat{\phi}\right)}{\sum_{j=1}^{n}\exp\left(-Y_j/\hat{\phi}\right)} \tag{3.12}$$

   by iteration then

$$\hat{\eta} = -\hat{\phi}\ln\left\{\frac{1}{n}\sum_{j=1}^{n}\exp\left(-Y_j/\hat{\phi}\right)\right\}. \tag{3.13}$$

4. Calculate $Z_i = F\left(Y_{(i)};\hat{\eta},\hat{\phi}\right)$ for $i = 1, 2, \ldots, n$ where F is specified in (3.2.4).

5. Calculate the statistics (3.8) and (3.14).

6. Modify the test statistic as in Table 3.2 using the modification for percentage points. If the statistic exceeds the value in the upper tail given at level α, H_0 is rejected at significance level α.

Similarly as (3.10) we can compute the Anderson-Darling statistic for extreme-value distribution. It has the form

$$\begin{aligned}
A^2 :=& n\int_{-\infty}^{\infty} \frac{\left[F_n(y) - \exp\left(-\exp\left[-\frac{y-\eta}{\phi}\right]\right)\right]^2}{\exp\left(-\exp\left[-\frac{y-\eta}{\phi}\right]\right)\left[1-\exp\left(-\exp\left[-\frac{y-\eta}{\phi}\right]\right)\right]} \\
&\frac{1}{\phi}\exp\left(-\exp\left[-\frac{y-\eta}{\phi}\right]\right)\exp\left[-\frac{y-\eta}{\phi}\right]\,dy \\
=&\frac{n}{\phi}\left[\int_{-\infty}^{Y_{(1)}} \frac{\left[\exp\left(-\exp\left[-\frac{y-\eta}{\phi}\right]\right)\right]^2}{\left[1-\exp\left(-\exp\left[-\frac{y-\eta}{\phi}\right]\right)\right]}\exp\left[-\frac{y-\eta}{\phi}\right]\,dy+\right. \\
&+\sum_{i=1}^{n-1}\int_{Y_{(i)}}^{Y_{(i+1)}} \frac{\left[\frac{i}{n}-\exp\left(-\exp\left[-\frac{y-\eta}{\phi}\right]\right)\right]^2}{\left[1-\exp\left(-\exp\left[-\frac{y-\eta}{\phi}\right]\right)\right]}\exp\left[-\frac{y-\eta}{\phi}\right]\,dy+ \\
&\left.+\sum_{i=1}^{n-1}\int_{Y_{(i+1)}}^{\infty}\left[1-\exp\left(-\exp\left[-\frac{y-\eta}{\phi}\right]\right)\right]\exp\left[-\frac{y-\eta}{\phi}\right]\,dy\right]
\end{aligned} \tag{3.14}$$

Table 3.2: Modifications and upper percentage points for the type-I extreme value distribution of the maximum [13]

Statistic		Significance level α			
		0,10	0,05	0,025	0,01
Statistic	Modification	Upper tail percentage points			
A^2	$A^2\,(1+0,2/\sqrt{n})$	0,637	0,757	0,877	1,038
Statistic	n	Upper tail percentage points			
$\sqrt{n}$D	10	0,760	0,819	0,880	0,944
	20	0,779	0,843	0,907	0,973
	50	0,790	0,856	0,922	0,988
	∞	0,803	0,874	0,939	1,007

Chapter 4

One-Way Analysis of Variance Type Models

First, we derive specific shapes of the log-likelihood equations of the parameter estimates and the Fisher information Matrix. Then attention is devoted to asymptotic tests on unknown parameters.

4.1 MLE for Two-Parameter Weibull Distribution

In the following test the models where one of the parameter of Weibull distribution might different are considered.

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ be independent and identically distributed such that $\mathbf{X}_i = (X_{i1}, \ldots, X_{in})^T$, for $i = 1, \ldots, k$, be a random sample from the two-parameter Weibull distribution with parameters α and β. We use the notation

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$$

and

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)\,.$$

For observation x_{ij} of the random variable X_{ij}, the likelihood element is defined as

$$L_{ij}\left(\alpha_i, \beta_i\right) := f\left(x_{ij}, \alpha_i, \beta_i\right), \tag{4.1}$$

where $f\left(x_{ij}, \alpha_i, \beta_i\right)$ is the density function from (1.3).

The *likelihood function* is

$$L\left(\boldsymbol{\alpha}, \boldsymbol{\beta}\right) = \prod_{i=1}^{k} \prod_{j=1}^{n} L_{ij}\left(\alpha_i, \beta_i\right). \tag{4.2}$$

For finding MLE it is more conveion to use the log-likelihood function which has a definition

$$\mathcal{L}\left(\boldsymbol{\alpha}, \boldsymbol{\beta}\right) := \ln L\left(\boldsymbol{\alpha}, \boldsymbol{\beta}\right) = \sum_{i=1}^{k} \sum_{j=1}^{n} \ln L_{ij}\left(\alpha_i, \beta_i\right). \tag{4.3}$$

The log-likelihood function for the standard Weibull model has the form

$$\mathcal{L}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{i=1}^{k}\sum_{j=1}^{n}\ln\left[\frac{\beta_i x_{ij}^{\beta_i-1}}{\alpha_i^{\beta_i}}\ \exp\left[-\left(\frac{x_{ij}}{\alpha_i}\right)^{\beta_i}\right]\right]$$
$$= \sum_{i=1}^{k}\sum_{j=1}^{n}\left[\ln\beta_i - \beta_i\ln\alpha_i + (\beta_i-1)\ln x_{ij} - \left(\frac{x_{ij}}{\alpha_i}\right)^{\beta_i}\right]. \qquad (4.4)$$

4.2 Model with Constant Shape Parameter

We define the parameter $\boldsymbol{\beta}$ such that

$$\beta_i = \beta,\ \text{for } i = 1,\dots,k.$$

We define parametrization of parameter $\boldsymbol{\alpha}$ in the form

$$\alpha_i = \xi + \delta_i,\ \text{for } i = 1,\dots,k$$

such that $\delta_1 = 0$ or $\sum_{i=1}^{k}\delta_i = 0$. We choose the first condition, then $\mathcal{L}(\boldsymbol{\alpha},\beta)$ has after the substitution form $\mathcal{L}(\xi,\delta_2,\dots,\delta_k,\beta)$. We define notation of parameters δ_i for $i = 2,\dots,k$ such that $\boldsymbol{\delta} = (\delta_2,\dots,\delta_k)$. Hereafter all derivatives with respect to δ_i are meant for $i \neq 1$.

The log-likelihood function for the standard Weibull model has the form

$$\mathcal{L}(\boldsymbol{\alpha},\beta) = \sum_{i=1}^{k}\sum_{j=1}^{n}\ln\left[\frac{\beta x_{ij}^{\beta-1}}{(\xi+\delta_i)^{\beta}}\ \exp\left[-\left(\frac{x_{ij}}{\xi+\delta_i}\right)^{\beta}\right]\right]$$
$$= \sum_{i=1}^{k}\sum_{j=1}^{n}\left[\ln\beta - \beta\ln(\xi+\delta_i) + (\beta-1)\ln x_{ij} - \left(\frac{x_{ij}}{\xi+\delta_i}\right)^{\beta}\right]$$
$$= nk\ln\beta - n\beta\sum_{i=1}^{k}\ln(\xi+\delta_i) + (\beta-1)\sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij} - \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{x_{ij}}{\xi+\delta_i}\right)^{\beta}. \qquad (4.5)$$

By solving the system of log-likelihood equations which has the form

$$\frac{\partial\mathcal{L}(\xi,\boldsymbol{\delta},\beta)}{\partial\xi} = -n\beta\sum_{i=1}^{k}\frac{1}{\xi+\delta_i} + \beta\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\beta}(\xi+\delta_i)^{-(\beta+1)} = 0, \qquad (4.6)$$

$$\frac{\partial\mathcal{L}(\xi,\boldsymbol{\delta},\beta)}{\partial\delta_i} = -\frac{n\beta}{\xi+\delta_i} + \beta\sum_{j=1}^{n}x_{ij}^{\beta}(\xi+\delta_i)^{-(\beta+1)} = 0, \qquad (4.7)$$

$$\frac{\partial\mathcal{L}(\xi,\boldsymbol{\delta},\beta)}{\partial\beta} = \frac{nk}{\beta} - n\sum_{i=1}^{k}\ln(\xi+\delta_i) + \sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij} - \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{x_{ij}}{\xi+\delta_i}\right)^{\beta}\ln\left(\frac{x_{ij}}{\xi+\delta_i}\right) = 0. \qquad (4.8)$$

we obtain MLE.

From (4.7) we derive

$$-\frac{n}{\xi+\delta_i} + (\xi+\delta_i)^{-(\beta+1)}\sum_{j=1}^{n}x_{ij}^{\beta} = 0$$

$$(\xi+\delta_i)^{-(\beta+1)}\sum_{j=1}^{n}x_{ij}^{\beta} = \frac{n}{\xi+\delta_i} \qquad (4.9)$$

After substitution (4.9) into the equation (4.6) we obtain

$$
\begin{aligned}
-n\sum_{i=1}^{k}\frac{1}{\xi+\delta_i}+\sum_{i=1}^{k}(\xi+\delta_i)^{-(\beta+1)}\sum_{j=1}^{n}x_{ij}^{\beta}&=0\\
-n\sum_{i=1}^{k}\frac{1}{\xi+\delta_i}+n\sum_{i=2}^{k}\frac{1}{\xi+\delta_i}+\xi^{-(\beta+1)}\sum_{j=1}^{n}x_{1j}^{\beta}&=0\\
-n\sum_{i=1}^{k}\frac{1}{\xi+\delta_i}+n\sum_{i=1}^{k}\frac{1}{\xi+\delta_i}-\frac{n}{\xi}+\xi^{-(\beta+1)}\sum_{j=1}^{n}x_{1j}^{\beta}&=0\\
\xi^{-(\beta+1)}\sum_{j=1}^{n}x_{1j}^{\beta}&=\frac{n}{\xi}\\
\xi^{-\beta}\sum_{j=1}^{n}x_{1j}^{\beta}&=n\\
\xi&=\left[\frac{1}{n}\sum_{j=1}^{n}x_{1j}^{\beta}\right]^{1/\beta} \qquad (4.10)
\end{aligned}
$$

From (4.9) we derive parameter δ_i such as

$$
\begin{aligned}
(\xi+\delta_i)^{-\beta}\sum_{j=1}^{n}x_{ij}^{\beta}&=n\\
(\xi+\delta_i)^{-\beta}&=\frac{n}{\sum_{j=1}^{n}x_{ij}^{\beta}} \qquad (4.11)\\
\delta_i&=\left[\frac{1}{n}\sum_{j=1}^{n}x_{ij}^{\beta}\right]^{1/\beta}-\left[\frac{1}{n}\sum_{j=1}^{n}x_{1j}^{\beta}\right]^{1/\beta}. \qquad (4.12)
\end{aligned}
$$

After substitution (4.9) and (4.11), the equation (4.8) has the form of

$$
\begin{aligned}
&\frac{nk}{\beta}-n\sum_{i=1}^{k}\ln(\xi+\delta_i)+\sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij}-\sum_{i=1}^{k}(\xi+\delta_i)^{-\beta}\sum_{j=1}^{n}x_{ij}^{\beta}\ln x_{ij}+\\
&\qquad+\sum_{i=1}^{k}\ln(\xi+\delta_i)(\xi+\delta_i)^{-\beta}\sum_{j=1}^{n}x_{ij}^{\beta}=0\\
&\frac{nk}{\beta}-n\sum_{i=1}^{k}\ln(\xi+\delta_i)+\sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij}-\sum_{i=1}^{k}\frac{n}{\sum_{j=1}^{n}x_{ij}^{\beta}}\sum_{j=1}^{n}x_{ij}^{\beta}\ln x_{ij}+\\
&\qquad+\sum_{i=1}^{k}\ln(\xi+\delta_i)(\xi+\delta_i)\frac{n}{\xi+\delta_i}=0\\
&\frac{k}{\beta}+\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij}-\sum_{i=1}^{k}\frac{\sum_{j=1}^{n}x_{ij}^{\beta}\ln x_{ij}}{\sum_{j=1}^{n}x_{ij}^{\beta}}=0. \qquad (4.13)
\end{aligned}
$$

After substitution (4.10) into the last part of right side of the equation (4.5) we obtain

$$\sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{x_{ij}}{\xi+\delta_i}\right)^{\beta}=\sum_{i=1}^{k}\left(\xi+\delta_i\right)^{-\beta}\sum_{j=1}^{n}x_{ij}^{\beta}=\sum_{i=1}^{k}\left(\xi+\delta_i\right)\left(\xi+\delta_i\right)^{-(\beta+1)}\sum_{j=1}^{n}x_{ij}^{\beta}$$

$$=\sum_{i=1}^{k}\left(\xi+\delta_i\right)\left(\frac{n}{\xi+\delta_i}\right)=nk. \tag{4.14}$$

We can derive a new form of the log-likelihood function after substitution (4.14) and (4.11) into (4.5) which we can use as the new possible way for optimization. The advantage for this way is that we optimize the equation with only one variable which is more easy than other methods.

$$\mathcal{L}\left(\beta\right)=nk\ln\beta-n\sum_{i=1}^{k}\ln\left[\frac{1}{n}\sum_{j=1}^{n}x_{ij}^{\beta}\right]+\left(\beta-1\right)\sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij}-nk$$

$$=nk\left[\ln\beta-1\right]-n\sum_{i=1}^{k}\ln\left[\frac{1}{n}\sum_{j=1}^{n}x_{ij}^{\beta}\right]+\left(\beta-1\right)\sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij} \tag{4.15}$$

The second derivations of the log-likelihood function are

$$\frac{\partial^2\mathcal{L}\left(\xi,\boldsymbol{\delta},\beta\right)}{\partial\xi^2}=n\beta\sum_{i=1}^{k}\frac{1}{\left(\xi+\delta_i\right)^2}-\beta\left(\beta+1\right)\sum_{i=1}^{k}\left(\xi+\delta_i\right)^{-(\beta+2)}\sum_{j=1}^{n}x_{ij}^{\beta}, \tag{4.16}$$

$$\frac{\partial^2\mathcal{L}\left(\xi,\boldsymbol{\delta},\beta\right)}{\partial\xi\partial\delta_i}=\frac{n\beta}{\left(\xi+\delta_i\right)^2}-\beta\left(\beta+1\right)\left(\xi+\delta_i\right)^{-(\beta+2)}\sum_{j=1}^{n}x_{ij}^{\beta}, \tag{4.17}$$

$$\frac{\partial^2\mathcal{L}\left(\xi,\boldsymbol{\delta},\beta\right)}{\partial\xi\partial\beta}=-n\sum_{i=1}^{k}\frac{1}{\xi+\delta_i}+\sum_{i=1}^{k}\left(\xi+\delta_i\right)^{-(\beta+1)}\sum_{j=1}^{n}x_{ij}^{\beta}-\beta\sum_{i=1}^{k}\ln\left(\xi+\delta_i\right)\left(\xi+\delta_i\right)^{-(\beta+1)}$$

$$\sum_{j=1}^{n}x_{ij}^{\beta}+\beta\sum_{i=1}^{k}\left(\xi+\delta_i\right)^{-(\beta+1)}\sum_{j=1}^{n}\ln x_{ij}x_{ij}^{\beta} \tag{4.18}$$

$$\frac{\partial^2\mathcal{L}\left(\xi,\boldsymbol{\delta},\beta\right)}{\partial\delta_i^2}=\frac{n\beta}{\left(\xi+\delta_i\right)^2}-\beta\left(\beta+1\right)\left(\xi+\delta_i\right)^{-(\beta+2)}\sum_{j=1}^{n}x_{ij}^{\beta}, \tag{4.19}$$

$$\frac{\partial^2\mathcal{L}\left(\xi,\boldsymbol{\delta},\beta\right)}{\partial\delta_i\partial\delta_j}=0,\ \text{for } i\neq j, \tag{4.20}$$

$$\frac{\partial^2\mathcal{L}\left(\xi,\boldsymbol{\delta},\beta\right)}{\partial\delta_i\partial\beta}=-\frac{n}{\xi+\delta_i}+\left(\xi+\delta_i\right)^{-(\beta+1)}\sum_{j=1}^{n}x_{ij}^{\beta}-\beta\ln\left(\xi+\delta_i\right)\left(\xi+\delta_i\right)^{-(\beta+1)}\sum_{j=1}^{n}x_{ij}^{\beta}+$$

$$+\beta\left(\xi+\delta_i\right)^{-(\beta+1)}\sum_{j=1}^{n}\ln x_{ij}x_{ij}^{\beta}. \tag{4.21}$$

$$\frac{\partial^2\mathcal{L}\left(\xi,\boldsymbol{\delta},\beta\right)}{\partial\beta^2}=-\frac{nk}{\beta^2}-\sum_{i=1}^{k}\left(\xi+\delta_i\right)^{-\beta}\sum_{j=1}^{n}\ln\left(\frac{x_{ij}}{\xi+\delta_i}\right)x_{ij}^{\beta}\ln x_{ij}+$$

$$+\sum_{i=1}^{k}\left(\xi+\delta_i\right)^{-\beta}\ln\left(\xi+\delta_i\right)\sum_{j=1}^{n}\ln\left(\frac{x_{ij}}{\xi+\delta_i}\right)x_{ij}^{\beta}, \tag{4.22}$$

for $i=2,\ldots,k$.

The Fisher information matrix $\mathbf{I}_n(\xi,\boldsymbol{\delta},\beta)$ of $\mathcal{L}(\xi,\boldsymbol{\delta},\beta)$ has the form of

$$
\begin{aligned}
\mathbf{I}_n(\xi,\boldsymbol{\delta},\beta)_{11} &= -E\left[\frac{\partial^2\mathcal{L}(\xi,\boldsymbol{\delta},\beta)}{\partial\xi^2}\right] = -n\beta\sum_{i=1}^{k}\frac{1}{(\xi+\delta_i)^2}+\sum_{i=1}^{k}\frac{\beta(\beta+1)}{(\xi+\delta_i)^{\beta+2}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\right]\\
&= n\beta^2\sum_{i=1}^{k}\frac{1}{(\xi+\delta_i)^2}\\
\mathbf{I}_n(\xi,\boldsymbol{\delta},\beta)_{1i} &= -E\left[\frac{\partial^2\mathcal{L}(\xi,\boldsymbol{\delta},\beta)}{\partial\xi\partial\delta_i}\right] = -\frac{n\beta}{(\xi+\delta_i)^2}+\frac{\beta(\beta+1)}{(\xi+\delta_i)^{\beta+2}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\right]\\
&= \frac{n\beta^2}{(\xi+\delta_i)^2}\\
\mathbf{I}_n(\xi,\boldsymbol{\delta},\beta)_{1(k+1)} &= -E\left[\frac{\partial^2\mathcal{L}(\xi,\boldsymbol{\delta},\beta)}{\partial\xi\partial\beta}\right]\\
&= \sum_{i=1}^{k}\frac{n}{\xi+\delta_i}-\sum_{i=1}^{k}\frac{1}{(\xi+\delta_i)^{\beta+1}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\right]+\beta\sum_{i=1}^{k}\frac{\ln(\xi+\delta_i)}{(\xi+\delta_i)^{\beta+1}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\right]\\
&-\sum_{i=1}^{k}\frac{\beta}{(\xi+\delta_i)^{\beta+1}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\ln X_{ij}\right]\\
&= \beta n\sum_{i=1}^{k}\frac{\ln(\xi+\delta_i)}{(\xi+\delta_i)}-n\sum_{i=1}^{k}\frac{1}{\xi+\delta_i}\left(\Gamma'(2)+\beta\ln(\xi+\delta_i)\right)\\
\mathbf{I}_n(\xi,\boldsymbol{\delta},\beta)_{ii} &= -E\left[\frac{\partial^2\mathcal{L}(\xi,\boldsymbol{\delta},\beta)}{\partial\delta_i^2}\right] = -\frac{n\beta}{(\xi+\delta_i)^2}+\frac{\beta(\beta+1)}{(\xi+\delta_i)^{\beta+2}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\right] = \frac{n\beta^2}{(\xi+\delta_i)^2}\\
\mathbf{I}_n(\xi,\boldsymbol{\delta},\beta)_{i(k+1)} &= -E\left[\frac{\partial^2\mathcal{L}(\xi,\boldsymbol{\delta},\beta)}{\partial\delta_i\partial\beta}\right]\\
&= \frac{n}{\xi+\delta_i}-\frac{1}{(\xi+\delta_i)^{\beta+1}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\right]+\beta\frac{\ln(\xi+\delta_i)}{(\xi+\delta_i)^{\beta+1}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\right]\\
&-\frac{\beta}{(\xi+\delta_i)^{\beta+1}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\ln X_{ij}\right] = \beta n\frac{\ln(\xi+\delta_i)}{(\xi+\delta_i)}-\frac{n}{\xi+\delta_i}\left(\Gamma'(2)+\beta\ln(\xi+\delta_i)\right)\\
\mathbf{I}_n(\xi,\boldsymbol{\delta},\beta)_{(k+1)(k+1)} &= -E\left[\frac{\partial^2\mathcal{L}(\xi,\boldsymbol{\delta},\beta)}{\partial\beta^2}\right] = \frac{nk}{\beta^2}+\sum_{i=1}^{k}\frac{1}{(\xi+\delta_i)^{\beta}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}(\ln X_{ij})^2\right]\\
&-\sum_{i=1}^{k}\frac{\ln(\xi+\delta_i)}{(\xi+\delta_i)^{\beta}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\ln X_{ij}\right]+\sum_{i=1}^{k}\frac{(\ln(\xi+\delta_i))^2}{(\xi+\delta_i)^{-\beta}}\sum_{j=1}^{n}E\left[X_{ij}^{\beta}\right]\\
&= \frac{nk}{\beta^2}\left(\Gamma''(2)+1\right)+n\Gamma(2)\sum_{i=1}^{k}(\ln(\xi+\delta_i))^2-n\sum_{i=1}^{k}(\ln(\xi+\delta_i))^2\\
&= \frac{nk}{\beta^2}\left(\Gamma''(2)+1\right) \qquad (4.23)
\end{aligned}
$$

for $i=2,\ldots,k$. The matrix is symmetric and other elements of the Fisher matrix are equal to 0.

4.3 Model with Constant Scale Parameter

We define the parameter $\boldsymbol{\alpha}$ such that

$$\alpha_i = \alpha, \quad \text{for}\, i = 1, \ldots, k.$$

We define substitution of parameter $\boldsymbol{\beta}$ in the form

$$\beta_i = \nu + \epsilon_i, \quad \text{for}\, i = 1, \ldots, k$$

such that $\epsilon_1 = 0$ or $\sum_{i=1}^{k} \epsilon_i = 0$. We choose the first condition, then $\mathcal{L}(\alpha, \boldsymbol{\beta})$ has after the substitution form $\mathcal{L}(\alpha, \epsilon_2, \ldots, \epsilon_k, \nu)$. We define notation of parameters ϵ_i for $i = 2, \ldots, k$ such that $\boldsymbol{\epsilon} = (\epsilon_2, \ldots, \epsilon_k)$. Hereafter all derivatives with respect to ϵ_i are meant for $i \neq 1$.

The log-likelihood function for the standard Weibull model has the form of

$$\begin{aligned}
\mathcal{L}(\alpha, \boldsymbol{\beta}) =& \sum_{i=1}^{k} \sum_{j=1}^{n} \ln \left[\frac{(\nu + \epsilon_i)\, x_{ij}^{\nu+\epsilon_i-1}}{\alpha^{\nu+\epsilon_i}} \exp\left[- \left(\frac{x_{ij}}{\alpha} \right)^{\nu+\epsilon_i} \right] \right] \\
=& \sum_{i=1}^{k} \sum_{j=1}^{n} \left[\ln(\nu + \epsilon_i) - (\nu + \epsilon_i) \ln \alpha + (\nu + \epsilon_i - 1) \ln x_{ij} - \left(\frac{x_{ij}}{\alpha} \right)^{\nu+\epsilon_i} \right] \\
=& n \sum_{i=1}^{k} \ln(\nu + \epsilon_i) - n \ln \alpha \sum_{i=1}^{k} (\nu + \epsilon_i) + \sum_{i=1}^{k} \sum_{j=1}^{n} (\nu + \epsilon_i - 1) \ln x_{ij} \\
& - \sum_{i=1}^{k} \sum_{j=1}^{n} \left(\frac{x_{ij}}{\alpha} \right)^{\nu+\epsilon_i}.
\end{aligned} \tag{4.24}$$

By solving the system of the log-likelihood equations which has form

$$\frac{\partial \mathcal{L}(\alpha, \boldsymbol{\epsilon}, \nu)}{\partial \alpha} = -\frac{n}{\alpha} \sum_{i=1}^{k} (\nu + \epsilon_i) + \sum_{i=1}^{k} \sum_{j=1}^{n} (\nu + \epsilon_i)\, x_{ij}^{\nu+\epsilon_i} \alpha^{-(\nu+\epsilon_i+1)} = 0, \tag{4.25}$$

$$\frac{\partial \mathcal{L}(\alpha, \boldsymbol{\epsilon}, \nu)}{\partial \epsilon_i} = \frac{n}{\nu + \epsilon_i} + \sum_{j=1}^{n} \ln x_{ij} - n \ln \alpha - \sum_{j=1}^{n} \left(\frac{x_{ij}}{\alpha} \right)^{\nu+\epsilon_i} \ln \left(\frac{x_{ij}}{\alpha} \right) = 0, \tag{4.26}$$

$$\frac{\partial \mathcal{L}(\alpha, \boldsymbol{\epsilon}, \nu)}{\partial \nu} = n \sum_{i=1}^{k} \frac{1}{\nu + \epsilon_i} - nk \ln \alpha + \sum_{i=1}^{k} \sum_{j=1}^{n} \ln x_{ij} - \sum_{i=1}^{k} \sum_{j=1}^{n} \left(\frac{x_{ij}}{\alpha} \right)^{\nu+\epsilon_i} \ln \left(\frac{x_{ij}}{\alpha} \right) = 0. \tag{4.27}$$

we obtain MLE.

The second derivative of the log-likelihood function are

$$\frac{\partial^2 \mathcal{L}(\alpha, \boldsymbol{\epsilon}, \nu)}{\partial \alpha^2} = \frac{n}{\alpha^2} \sum_{i=1}^{k} (\nu + \epsilon_i) - \sum_{i=1}^{k} \sum_{j=1}^{n} (\nu + \epsilon_i)(\nu + \epsilon_i + 1) x_{ij}^{\nu+\epsilon_i} \alpha^{-(\nu+\epsilon_i+2)}, \tag{4.28}$$

$$\begin{aligned} \frac{\partial^2 \mathcal{L}(\alpha, \boldsymbol{\epsilon}, \nu)}{\partial \alpha \partial \epsilon_i} =& - \frac{n}{\alpha} + \sum_{j=1}^{n} x_{ij}^{\nu+\epsilon_i} \alpha^{-(\nu+\epsilon_i+1)} + \sum_{j=1}^{n} (\nu + \epsilon_i) x_{ij}^{\nu+\epsilon_i} \ln x_{ij} \alpha^{-(\nu+\epsilon_i+1)} \\ & - \sum_{j=1}^{n} (\nu + \epsilon_i) x_{ij}^{\nu+\epsilon_i} \ln \alpha \alpha^{-(\nu+\epsilon_i+1)}, \end{aligned} \tag{4.29}$$

$$\begin{aligned} \frac{\partial^2 \mathcal{L}(\alpha, \boldsymbol{\epsilon}, \nu)}{\partial \alpha \partial \nu} =& - \frac{nk}{\alpha} + \sum_{i=1}^{k} \sum_{j=1}^{n} x_{ij}^{\nu+\epsilon_i} \alpha^{-(\nu+\epsilon_i+1)} + \sum_{i=1}^{k} \sum_{j=1}^{n} (\nu + \epsilon_i) \ln x_{ij} x_{ij}^{\nu+\epsilon_i} \alpha^{-(\nu+\epsilon_i+1)} \\ & - \sum_{i=1}^{k} \sum_{j=1}^{n} (\nu + \epsilon_i) \ln \alpha x_{ij}^{\nu+\epsilon_i} \alpha^{-(\nu+\epsilon_i+1)}, \end{aligned} \tag{4.30}$$

$$\frac{\partial^2 \mathcal{L}(\alpha, \boldsymbol{\epsilon}, \nu)}{\partial \epsilon_i^2} = - \frac{n}{(\nu + \epsilon_i)^2} - \sum_{j=1}^{n} \left(\frac{x_{ij}}{\alpha} \right)^{\nu+\epsilon_i} \left[\ln \left(\frac{x_{ij}}{\alpha} \right) \right]^2, \tag{4.31}$$

$$\frac{\partial^2 \mathcal{L}(\alpha, \boldsymbol{\epsilon}, \nu)}{\partial \epsilon_i \partial \epsilon_j} = 0, \text{ for} i \neq j, \tag{4.32}$$

$$\frac{\partial^2 \mathcal{L}(\alpha, \boldsymbol{\epsilon}, \nu)}{\partial \epsilon_i \partial \nu} = - \frac{n}{(\nu + \epsilon_i)^2} - \sum_{j=1}^{n} \left(\frac{x_{ij}}{\alpha} \right)^{\nu+\epsilon_i} \left[\ln \left(\frac{x_{ij}}{\alpha} \right) \right]^2, \tag{4.33}$$

$$\frac{\partial^2 \mathcal{L}(\alpha, \boldsymbol{\epsilon}, \nu)}{\partial \nu^2} = - n \sum_{i=1}^{k} \frac{1}{(\nu + \epsilon_i)^2} - \sum_{i=1}^{k} \sum_{j=1}^{n} \left(\frac{x_{ij}}{\alpha} \right)^{\nu+\epsilon_i} \left[\ln \left(\frac{x_{ij}}{\alpha} \right) \right]^2, \tag{4.34}$$

for $i = 2, \ldots, k$.

The Fisher information matrix $\mathbf{I}_n(\alpha,\boldsymbol{\epsilon},\nu)$ of $\mathcal{L}(\alpha,\boldsymbol{\epsilon},\nu)$ has the form of

$$
\begin{aligned}
\mathbf{I}_n(\alpha,\boldsymbol{\epsilon},\nu)_{11} &= -E\left[\frac{\partial^2\mathcal{L}(\alpha,\boldsymbol{\epsilon},\nu)}{\partial\alpha^2}\right] = -\frac{n}{\alpha^2}\sum_{i=1}^{k}(\nu+\epsilon_i)+\sum_{i=1}^{k}\sum_{j=1}^{n}(\nu+\epsilon_i)(\nu+\epsilon_i+1)\,\alpha^{-(\nu+\epsilon_i+2)}\\
E\left[X_{ij}^{\nu+\epsilon_i}\right] &= \frac{n}{\alpha^2}\left[\sum_{i=1}^{k}(\nu+\epsilon_i)(\nu+\epsilon_i+1)-\sum_{i=1}^{k}(\nu+\epsilon_i)\right] = \frac{n}{\alpha^2}\sum_{i=1}^{k}(\nu+\epsilon_i)^2\\
\mathbf{I}_n(\alpha,\boldsymbol{\epsilon},\nu)_{1i} &= -E\left[\frac{\partial^2\mathcal{L}(\alpha,\boldsymbol{\epsilon},\nu)}{\partial\alpha\partial\epsilon_i}\right] = \frac{n}{\alpha}-\sum_{j=1}^{n}\alpha^{-(\nu+\epsilon_i+1)}E\left[X_{ij}^{\nu+\epsilon_i}\right]\\
&-\sum_{j=1}^{n}(\nu+\epsilon_i)\,\alpha^{-(\nu+\epsilon_i+1)}E\left[X_{ij}^{\nu+\epsilon_i}\ln X_{ij}\right]+\sum_{j=1}^{n}(\nu+\epsilon_i)\ln\alpha\alpha^{-(\nu+\epsilon_i+1)}\\
E\left[X_{ij}^{\nu+\epsilon_i}\right] &= \frac{n(\nu+\epsilon_i)\ln\alpha}{\alpha}-\frac{n}{\nu+\epsilon_i}\left(\Gamma'(2)+(\nu+\epsilon_i)\ln\alpha\right)\\
\mathbf{I}_n(\alpha,\boldsymbol{\epsilon},\nu)_{1(k+1)} &= -E\left[\frac{\partial^2\mathcal{L}(\alpha,\boldsymbol{\epsilon},\nu)}{\partial\alpha\partial\nu}\right]\\
&= \frac{nk}{\alpha}-\sum_{i=1}^{k}\sum_{j=1}^{n}\alpha^{-(\nu+\epsilon_i+1)}E\left[X_{ij}^{\nu+\epsilon_i}\right]-\sum_{i=1}^{k}\sum_{j=1}^{n}(\nu+\epsilon_i)\,\alpha^{-(\nu+\epsilon_i+1)}\\
E\left[X_{ij}^{\nu+\epsilon_i}\ln X_{ij}\right] &+ \sum_{i=1}^{k}\sum_{j=1}^{n}(\nu+\epsilon_i)\,\alpha^{-(\nu+\epsilon_i+1)}\ln\alpha E\left[X_{ij}^{\nu+\epsilon_i}\right]\\
&= -\frac{n}{\alpha}\sum_{i=1}^{k}\left(\Gamma'(2)+(\nu+\epsilon_i)\ln\alpha\right)+\frac{n}{\alpha}\ln\alpha\sum_{i=1}^{k}(\nu+\epsilon_i) = -\frac{nk}{\alpha}\Gamma'(2)\\
\mathbf{I}_n(\alpha,\boldsymbol{\epsilon},\nu)_{ii} &= -E\left[\frac{\partial^2\mathcal{L}(\alpha,\boldsymbol{\epsilon},\nu)}{\partial\epsilon_i^2}\right] = \frac{n}{(\nu+\epsilon_i)^2}+\sum_{j=1}^{n}\alpha^{-(\nu+\epsilon_i)}E\left[X_{ij}^{\nu+\epsilon_i}\left[\ln\left(\frac{X_{ij}}{\alpha}\right)\right]^2\right]\\
&= \frac{n}{(\nu+\epsilon_i)^2}+\frac{n}{(\nu+\epsilon_i)^2}\Gamma''(2)+\frac{2n}{\nu+\epsilon_i}\ln\alpha\Gamma'(2)+n(\ln\alpha)^2\Gamma(2)\\
&= \frac{n}{(\nu+\epsilon_i)^2}\left(\Gamma''(2)+1\right)+n\ln\alpha\left(\frac{2}{\nu+\epsilon_i}\Gamma'(2)+\ln\alpha\right)\\
\mathbf{I}_n(\alpha,\boldsymbol{\epsilon},\nu)_{i(k+1)} &= -E\left[\frac{\partial^2\mathcal{L}(\alpha,\boldsymbol{\epsilon},\nu)}{\partial\epsilon_i\partial\nu}\right] = \frac{n}{(\nu+\epsilon_i)^2}+\sum_{j=1}^{n}\alpha^{-(\nu+\epsilon_i)}E\left[X_{ij}^{\nu+\epsilon_i}\left[\ln\left(\frac{X_{ij}}{\alpha}\right)\right]^2\right]\\
&= \frac{n}{(\nu+\epsilon_i)^2}+\frac{n}{(\nu+\epsilon_i)^2}\Gamma''(2)+\frac{2n}{\nu+\epsilon_i}\ln\alpha\Gamma'(2)+n(\ln\alpha)^2\Gamma(2)\\
&= \frac{n}{(\nu+\epsilon_i)^2}\left(\Gamma''(2)+1\right)+n\ln\alpha\left(\frac{2}{\nu+\epsilon_i}\Gamma'(2)+\ln\alpha\right)\\
\mathbf{I}_n(\alpha,\boldsymbol{\epsilon},\nu)_{(k+1)(k+1)} &= -E\left[\frac{\partial^2\mathcal{L}(\alpha,\boldsymbol{\epsilon},\nu)}{\partial\nu^2}\right] = n\sum_{i=1}^{k}\frac{1}{(\nu+\epsilon_i)^2}+\sum_{i=1}^{k}\sum_{j=1}^{n}\alpha^{-(\nu+\epsilon_i)}E\left[X_{ij}^{\nu+\epsilon_i}\left[\ln\left(\frac{X_{ij}}{\alpha}\right)\right]^2\right]\\
&= n\sum_{i=1}^{k}\frac{1}{(\nu+\epsilon_i)^2}+n\Gamma''(2)\sum_{i=1}^{k}\frac{1}{(\nu+\epsilon_i)^2}+n\ln\alpha\Gamma'(2)\sum_{i=1}^{k}\frac{1}{\nu+\epsilon_i}+kn(\ln\alpha)^2\\
&= n\left(\Gamma''(2)+1\right)\sum_{i=1}^{k}\frac{1}{(\nu+\epsilon_i)^2}+n\ln\alpha\left(\Gamma'(2)\sum_{i=1}^{k}\frac{1}{\nu+\epsilon_i}+k\ln\alpha\right) \qquad (4.35)
\end{aligned}
$$

for $i=2,\ldots,k$. The matrix is symmetric and other elements of the Fisher matrix are equal to 0.

4.4 Model with Constant Parameters

We define the parameter α such that

$$\alpha_i = \xi, \text{ for } i = 1, \ldots, k$$

and the parameter β such that

$$\beta_i = \nu, \text{ for } i = 1, \ldots, k,$$

which mean that $\boldsymbol{\delta} = \mathbf{0}$ and $\boldsymbol{\epsilon} = \mathbf{0}$.

Therefore

$$\begin{aligned}\mathcal{L}(\xi, \nu) = \sum_{i=1}^{k}\sum_{j=1}^{n} \ln\left[\frac{\nu x_{ij}^{\nu-1}}{\xi^{\nu}} \exp\left[-\left(\frac{x_{ij}}{\xi}\right)^{\nu}\right]\right] = \\ = nk\ln\nu - nk\nu\ln\xi + (\nu-1)\sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij} - \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{x_{ij}}{\xi}\right)^{\nu}.\end{aligned} \tag{4.36}$$

The system of the log-likelihood equations is

$$\frac{\partial\mathcal{L}(\xi,\nu)}{\partial\xi} = -\frac{nk\nu}{\xi} + \nu\xi^{-(\nu+1)}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu} = 0, \tag{4.37}$$

$$\frac{\partial\mathcal{L}(\xi,\nu)}{\partial\nu} = \frac{nk}{\nu} - nk\ln\xi + \sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij} + \ln\xi\xi^{-\nu}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu} - \xi^{-\nu}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu}\ln x_{ij} = 0. \tag{4.38}$$

From (4.37) we derive

$$\xi^{-(\nu+1)}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu} = \frac{nk}{\xi}, \tag{4.39}$$

and

$$\begin{aligned}\xi^{-\nu}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu} &= nk \\ \xi^{\nu} &= \frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu} \\ \xi &= \left[\frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu}\right]^{1/\nu}\end{aligned} \tag{4.40}$$

After substitution (4.39) into the last part of right side of the equation (4.5) we obtain

$$\sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{x_{ij}}{\xi}\right)^{\nu} = \xi^{-\nu}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu} = \xi\xi^{-(\nu+1)}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu} = \xi\frac{nk}{\xi} = nk. \tag{4.41}$$

After substitution (4.39), the equation (4.38) has the form of

$$\frac{nk}{\nu} - nk\ln\xi + \sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij} + \xi\frac{nk}{\xi}\ln\xi\xi - \left[\frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu}\right]^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu}\ln x_{ij} = 0$$

$$\frac{1}{\nu} + \frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij} - \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu}\ln x_{ij}}{\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu}} = 0. \tag{4.42}$$

We can derive a new form of the log-likelihood function (4.36)

$$\begin{aligned}\mathcal{L}(\nu) &= nk\ln\nu - nk\nu\frac{1}{\nu}\ln\left[\frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu}\right] + (\nu-1)\sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij} - nk \\ &= nk\left[\ln\nu - 1\right] - nk\ln\left[\frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}^{\nu}\right] + (\nu-1)\sum_{i=1}^{k}\sum_{j=1}^{n}\ln x_{ij}.\end{aligned} \tag{4.43}$$

4.5 Tests with Nuisance Parameters

The primary source of this section was the book [2].

Let $\boldsymbol{\theta}$ be $m-$dimensional parameter from the parameter space $\Omega \in \mathcal{R}^m$ where $m \geq 2$. Let $0 \leq k < m$. Label

$$\boldsymbol{\tau} = (\theta_1, \dots\ \theta_k)^T, \text{ and } \boldsymbol{\psi} = (\theta_{k+1}, \dots\ \theta_m)^T. \tag{4.44}$$

Then $\boldsymbol{\theta} = \left(\boldsymbol{\tau}^T, \boldsymbol{\psi}^T\right)^T$. We denote the parameters from vector $\boldsymbol{\psi}$ as the *nuisance parameters.*

We need to introduce terms the score and partitioned Fisher information matrix $\mathbf{I}_n(\boldsymbol{\theta})$ for computing an inverse matrix.

Let

$$U_1(\boldsymbol{\theta}) = \left(\frac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\theta_1}, \dots, \frac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\theta_k}\right)^T, \mathbf{U}_2(\boldsymbol{\theta}) = \left(\frac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\theta_{k+1}}, \dots, \frac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\theta_m}\right)^T \tag{4.45}$$

then we called vectors $\mathbf{U}_1(\boldsymbol{\theta})$ and $\mathbf{U}_2(\boldsymbol{\theta})$ as the *score.*

The matrix $\mathbf{I}_n(\boldsymbol{\theta})$ partitioned into four blocks has form

$$\mathbf{I}_n(\boldsymbol{\theta}) = \begin{pmatrix} \mathbf{I}_n(\boldsymbol{\theta})_{11} & \mathbf{I}_n(\boldsymbol{\theta})_{12} \\ \mathbf{I}_n(\boldsymbol{\theta})_{21} & \mathbf{I}_n(\boldsymbol{\theta})_{22} \end{pmatrix},$$

where $\mathbf{I}_n(\boldsymbol{\theta})_{11}$ and $\mathbf{I}_n(\boldsymbol{\theta})_{22}$ are matrices with dimension $k \times k$, and $(m-k) \times (m-k)$ respectively.

Lemma 4.1. *Let the Fisher information matrix*

$$\mathbf{I}_n = \begin{pmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} \\ \mathbf{I}_{21} & \mathbf{I}_{22} \end{pmatrix},$$

be a regular matrix such that $\mathbf{I}_{11}$ *and* $\mathbf{I}_{22}$ *are regular square matrices.*

$$\begin{array}{lll} \mathbf{I}_{11.2} = \mathbf{I}_{11} - \mathbf{I}_{12}\mathbf{I}_{22}^{-1}\mathbf{I}_{21}, & \mathbf{I}^{11} = \mathbf{I}_{11.2}^{-1}, & \mathbf{I}^{12} = -\mathbf{I}_{11.2}^{-1}\mathbf{I}_{12}\mathbf{I}_{22}^{-1}, \\ \mathbf{I}_{22.1} = \mathbf{I}_{22} - \mathbf{I}_{21}\mathbf{I}_{11}^{-1}\mathbf{I}_{12}, & \mathbf{I}^{22} = \mathbf{I}_{22.1}^{-1}, & \mathbf{I}^{21} = -\mathbf{I}_{22.1}^{-1}\mathbf{I}_{21}\mathbf{I}_{11}^{-1}. \end{array}$$

Then

$$\mathbf{I}_n^{-1} = \begin{pmatrix} \mathbf{I}^{11} & \mathbf{I}^{12} \\ \mathbf{I}^{21} & \mathbf{I}^{22} \end{pmatrix}.$$

Proof. The proof omitted. We refer to [2]. □

We denote $\hat{\boldsymbol{\theta}}_n = \left(\hat{\boldsymbol{\tau}}_n^T, \hat{\boldsymbol{\psi}}_n^T\right)^T$ the vector of MLE of parameters. The following 3 tests can be apply when testing the null hypothesis

$$H_0 : \boldsymbol{\tau} = \boldsymbol{\tau}_0.$$

We denote $\tilde{\boldsymbol{\theta}}_n = \left(\boldsymbol{\tau}_0^T, \tilde{\boldsymbol{\psi}}_n^T\right)^T$.

4.5.1 Score Test

Theorem 4.2. *[2] Let all conditions from the theorem 2.4 are valid for $\tilde{\boldsymbol{\theta}}_n$. If the part of Fisher matrix $\mathbf{I}_{11.2}\left(\tilde{\boldsymbol{\theta}}_n\right)$ is continuous function in point $\boldsymbol{\theta}_0$, then*

$$S\left(\tilde{\boldsymbol{\theta}}_n\right) = \left[\mathbf{U}_1\left(\tilde{\boldsymbol{\theta}}_n\right)\right]^T \left[\mathbf{I}_{11.2}\left(\tilde{\boldsymbol{\theta}}_n\right)\right]^{-1} \mathbf{U}_1\left(\tilde{\boldsymbol{\theta}}_n\right) \tag{4.46}$$

has the asymptotic χ_k^2 distribution under the hypothesis.

We reject the null hypotheses if

$$S\left(\tilde{\boldsymbol{\theta}}_n\right) \geq \chi_k^2\left(\alpha\right), \tag{4.47}$$

where k is equal to dimension of $\boldsymbol{\tau}$ and α is the significance level.

4.5.2 Wald Test

Theorem 4.3. *[2] Let all conditions from the theorem 2.4 are valid for $\tilde{\boldsymbol{\theta}}_n$. If the part of Fisher matrix $\mathbf{I}_{11.2}\left(\tilde{\boldsymbol{\theta}}_n\right)$ is continuous function in point $\boldsymbol{\theta}_0$, then*

$$W\left(\tilde{\boldsymbol{\theta}}_n\right) = \left(\hat{\boldsymbol{\tau}}_n - \boldsymbol{\tau}_0\right)^T \mathbf{I}_{11.2}\left(\hat{\boldsymbol{\theta}}_n\right)\left(\hat{\boldsymbol{\tau}}_n - \boldsymbol{\tau}_0\right) \tag{4.48}$$

has the asymptotic χ_k^2 distribution under the hypothesis.

We reject the null hypotheses if

$$W\left(\tilde{\boldsymbol{\theta}}_n\right) \geq \chi_k^2\left(\alpha\right), \tag{4.49}$$

where k is equal to dimension of $\boldsymbol{\tau}$ and α is the significance level.

4.5.3 Likelihood Ratio Test

Theorem 4.4. *[2] Let all conditions from the theorem 2.4 are valid for $\tilde{\boldsymbol{\theta}}_n$. If the part of Fisher matrix $\mathbf{I}_{11.2}\left(\tilde{\boldsymbol{\theta}}_n\right)$ is continuous function in point $\boldsymbol{\theta}_0$, then*

$$LR\left(\tilde{\boldsymbol{\theta}}_n\right) = 2\left[\mathcal{L}\left(\hat{\boldsymbol{\theta}}_n\right) - \mathcal{L}\left(\tilde{\boldsymbol{\theta}}_n\right)\right] \tag{4.50}$$

has the asymptotic χ_k^2 distribution under the hypothesis.

We reject the null hypotheses if

$$LR\left(\tilde{\boldsymbol{\theta}}_n\right) \geq \chi_k^2\left(\alpha\right), \tag{4.51}$$

where k is equal to dimension of $\boldsymbol{\tau}$ and α is the significance level.

Chapter 5

Applications

In this chapter we confirm the properties of the method of estimation of parameters of two-parameter Weibull distribution and the properties of tests and we show the application of the methods and tests on the example. The theory of estimation methods for the two-parameters Weibull distribution, which was discussed in previous chapters, has been programmed to verify functionality and observed properties of the estimates. It was mainly used by coded in freeware software R and GAMS.

The program named Rkod_MT_Konecna.R is programmed in the R program (version R-3.2.0 for Windows). In this program you can use the simulated data matrix, dataSIM.csv and dataSIM2.csv. The parameters of the simulation data we can see in Table 5.1. All histograms presented in this chapter where are calculated from the data in data dataSIm2.csv. The histograms are for all 1000 the random samples.

Table 5.1: Parameters of simulation saved in csv files, namely the number of samples m, sample size n, scale parameter α and shape parameter β

The simulation data	m	n	α	β
dataSIM.csv	1000	30	1	50
dataSIM2.csv	1000	30	1	5

Table 5.2: The table of authors function for estimation of the parameters α and β in R

Name of method	Section	$\hat{\alpha}$	$\hat{\beta}$
weibull.moq	General case	(2.5)	(2.4)
	The sum of prob. equal 1	(2.30)	(2.29)
	Special quantiles	(2.31)	(2.32)
weibull.moq4	Method of Quantiles for intervals	(2.46)	(2.45)
weibull.mle1	Maximize the equation	(2.62)	(2.64)
weibull.mle2	Find root	(2.62)	(2.63)
weibull.WPP1	Rstudio function Find the Least Squares Fit		
weibull.WPP2	LRM	(2.70)	(2.67)

5.1 Application of the Method of Quantiles

The properties of the methods were evaluated from histograms and values of variances and the covariance. The theoretical values are found in Table 5.3 and values of simulated data in histograms in Figure 5.1. Histograms are represented by histogram (5.1) since looked similar. So we just put the histograms for the best configuration of method.

From histograms and parameters for each kind of method we can see, bast on that we can conclude the the method of quantiles for $p_1 = 0,23875930$ and $p_2 = 0,92656148$ from the article by Dubay [4] has the best properties.

Our results of minimization of determinant the asymptotic covariance matrix $\mathbf{\Sigma}$ from (2.11) are $p_1 = 0.2624487$ and $p_2 = 0.2624487$ which was computed in R by R function One Dimensional Optimization (optimize). The variances, the covariance and the determinant of the asymptotic covariance matrix have a similar values for this quantiles. The difference between (2.26) and (2.27) was probably caused by round error in R. This can be observed in Table 5.3 with the variance and covariance of the estimates. This implies that the general case with $p_1 = 0.2624487$ and $p_2 = 0.2624487$ is the best of the method of quantiles in the sense of the determinant of the asymptotic covariance matrix.

In Table 5.3 we can see the values of the variances, the covariance and the determinant of the asymptotic covariance matrix for the special quantiles. Here we present the results for $p_2 = 0,632$ and the value of p_1 our optimal choice and from [14]. The values of $\operatorname{Var}\hat{\alpha}$ and $\operatorname{Cov}(\hat{\alpha}, \hat{\beta})$ are dependent on the parameter β. The formulas (2.41), (2.40) and (2.42) are used in the Table 5.3.

In the table (5.3) there are computed the value of the variances, covariance and the determinant of the covariance matrix for the value of quantiles from sources and chosen.

Table 5.3: The table of variances and covariance for three kinds of the method of quantiles.

Quantiles	$\frac{n}{\beta^2}\operatorname{Var}\hat{\beta}$	$\frac{n\beta^2}{\alpha^2}\operatorname{Var}\hat{\alpha}$	$\frac{n}{\alpha}\operatorname{Cov}(\hat{\alpha},\hat{\beta})$	$\frac{n^2}{\alpha^2}\lvert\mathbf{\Sigma}\rvert$
$p_1 = 0.23875930$, $p_2 = 0.92656148$	1.016962	1.586453	0.3508805	1.490246
$p_1 = 0.2624487$, $p_2 = 0.9162927$	1.062666	1.523359	0.3418818	1.501938
$p_1 = 0.1$, $p_2 = 0.9$	1.134255	1.816171	0.5441425	1.76391
$p_1 = 0.1362754$, $p_2 = 0.8637246$	1.154364	1.587055	0.3919145	1.678442
$p_1 = 0.2$, $p_2 = 0.8$	1.325034	1.435762	0.2013188	1.861904
$p_1 = 0.3$, $p_2 = 0.7$	2.015187	1.492466	-0.1287286	2.991027
$p_1 = 0.4$, $p_2 = 0.6$	4.372474	1.928413	-1.002275	7.427381
$p_1 = 0.1342915$, $p_2 = 0.632$	1.873413	1.717391	-0.3313666	3.107579
$p_1 = 0.31$, $p_2 = 0.632$	2.60351	1.717391	-0.5110185	4.210105

We can see in Table 5.3 that for the general case, the method of quantiles is the best. The values for the quantiles from [4] are in the first row. The values for the quantiles which we optimized are in the second row of table (5.3). We can see that the values are very similar.

We can see in Table 5.3, that our quantile $p_1 = 0.1342915$ is better then the quantiles $p_1 = 0.31$ from [14] because $p_1 = 0.1342915$ has smaller the determinant of $\mathbf{\Sigma}$.

We can see from formulas (2.41), (2.40) and (2.42) that the determinant of the asymptotic covariance matrix for the special quantiles is independent on β. Our results of mineralization of the determinant the asymptotic covariance matrix $\mathbf{\Sigma}$ from (2.11) for the special quantiles are $p_1 = 0.1342915$ which was compute in R.

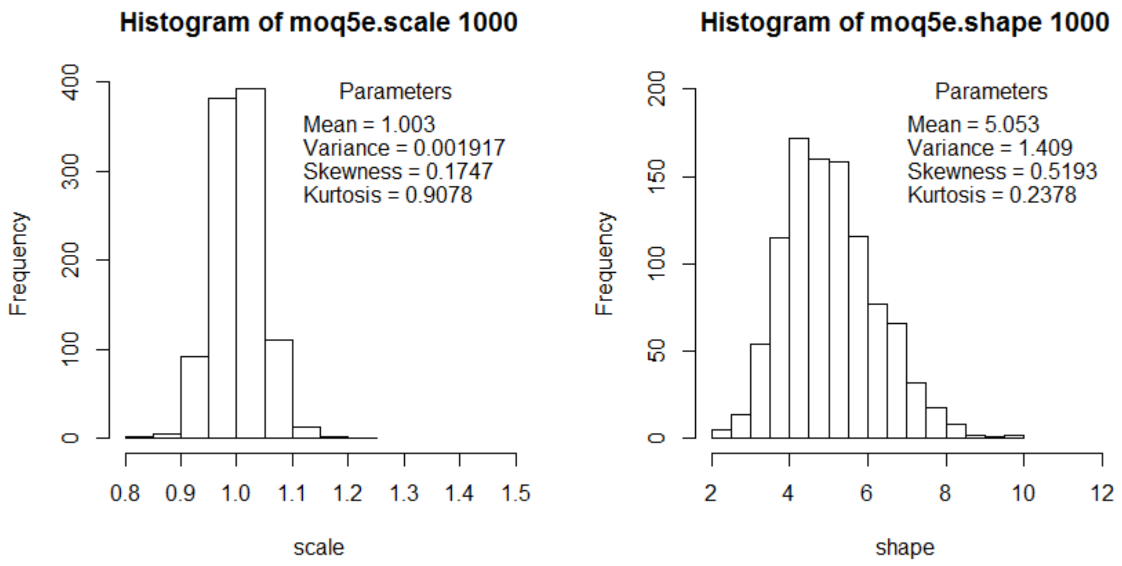


Figure 5.1: The histograms for estimation of parameters by the method of quantiles - the general case for 1000 random samples from dataSIM2.csv

5.2 Application of the Maximum Likelihood Estimations

In the first function weibull.mle1 for maximum likelihood estimation we optimize the log-likelihood function dependent only on parameter shape (2.64). We use function (2.64) and its gradient (2.63). The initial value for optimization uses the estimation of parameter shape by the method of quantiles - the general case.

In our opinion the program R is less useful for this complicated optimization. For example the program GAMS is better for optimization required by this kind of problem. Program GAMS is specialising for optimization and it has no problems with non-linear optimization.

The Fifures 5.2 and 5.2 represented the histograms of estimation of parameters by MLE, we can see that both methods is in an accordance with the asymptotic variance (2.60).

In the second function weibull.mle2 for maximum likelihood estimation we look for the positive root of the equation of the first derivative, the log-likelihood function dependent only on parameter shape (2.64).

The both methods have almost the same results.

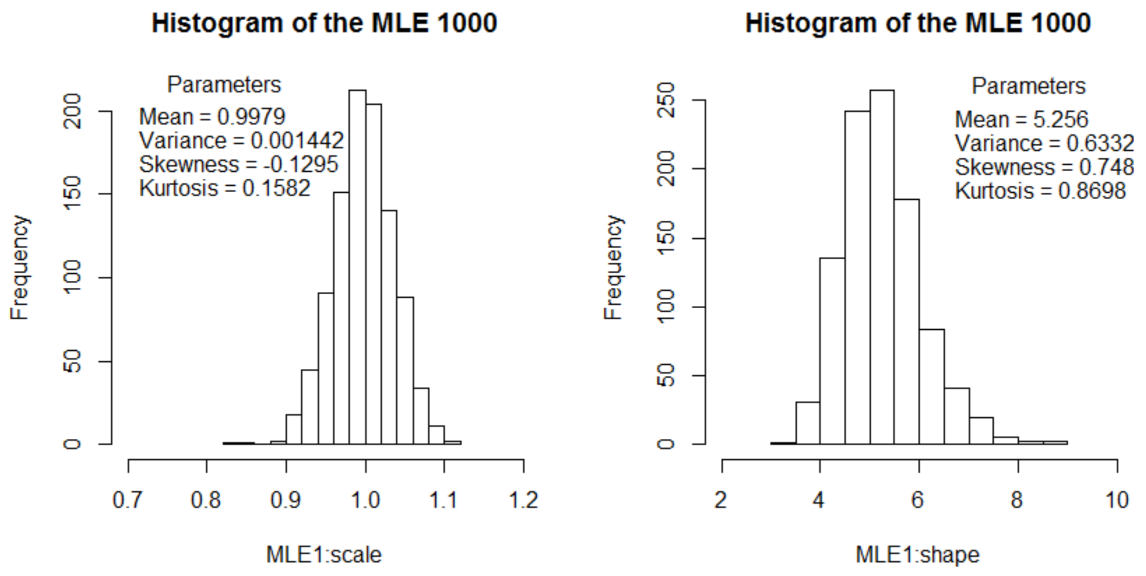


Figure 5.2: The histograms for estimation of parameters by MLE - the function weibull.mle1 on 1000 random samples from dataSIM2.csv

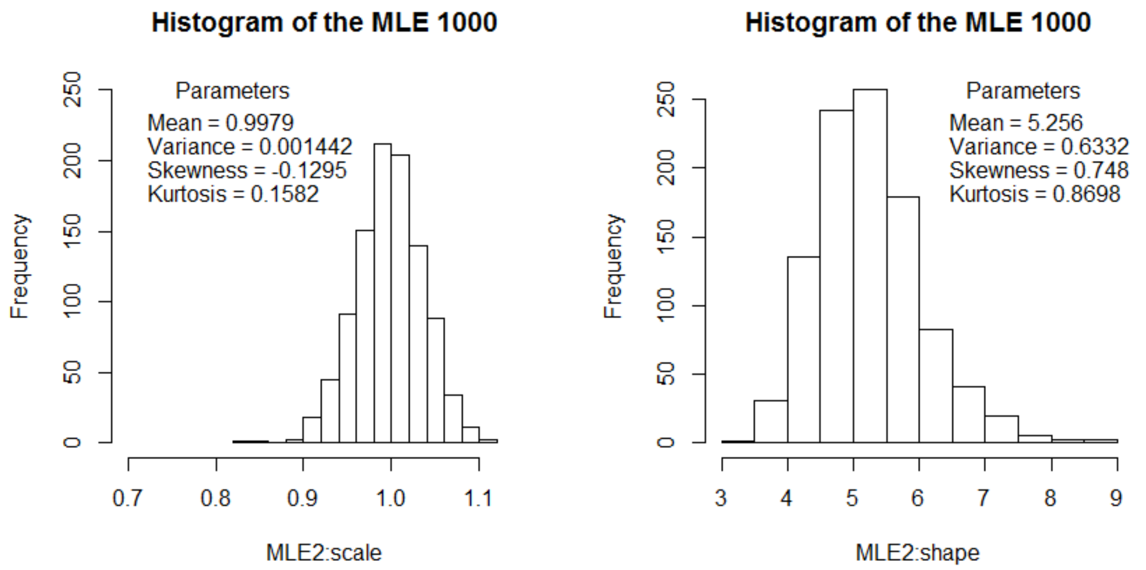


Figure 5.3: The histograms for estimation of parameters by MLE - the function weibull.mle2 on 1000 random samples from dataSIM2.csv

5.3 Application of Weibull probability plot

The graphical method of estimation of parameters are used in the two function weibull.WPP1 and weibull.WPP2. In the first function weibull.WPP1 we use the function of R Find the Least Squares Fit (lsfit). In the second function weibull.WPP2 we use two formulas for estimation of parameters β (2.67) and α (2.70). If we compare both methods, we deduce that both have the same estimations. Hence we present the graph and the histogram only for second function.

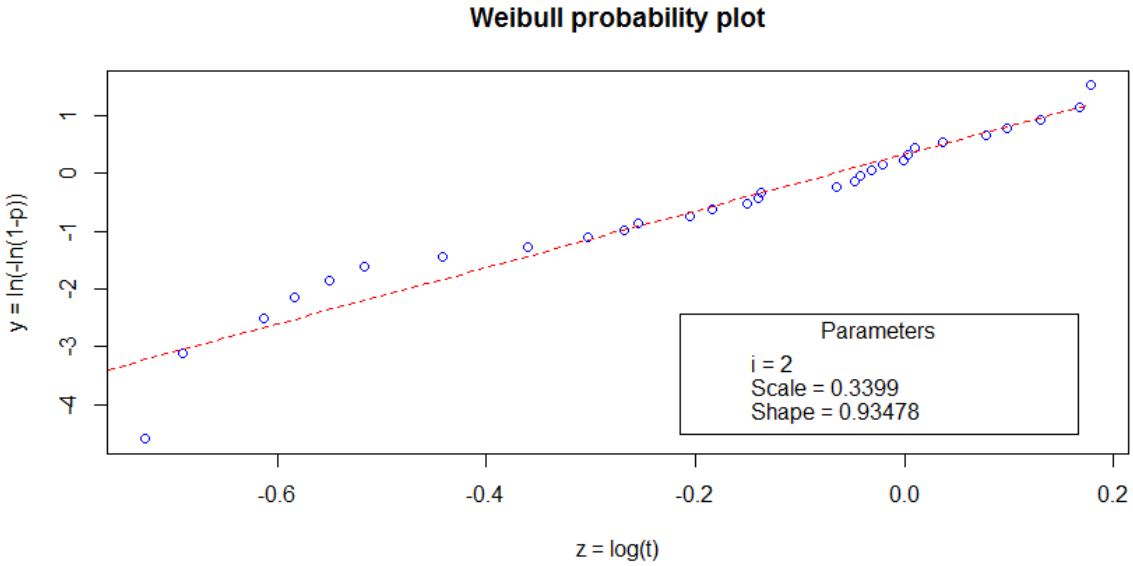
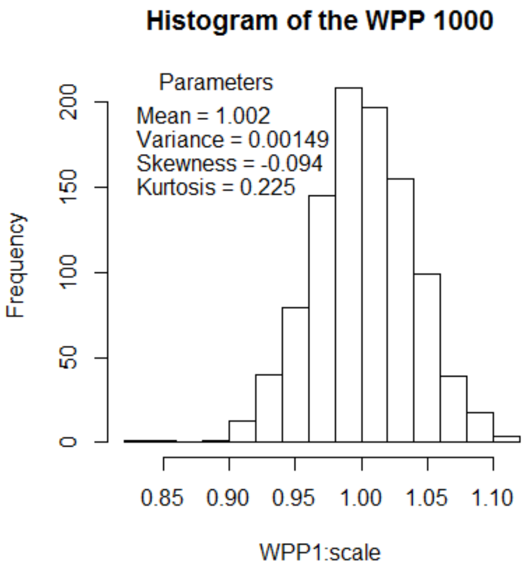


Figure 5.4: The Weibull probability plot for random sample $\mathbf{X}_2$ from dataSIM2.csv. In the box the index of sample in dataSIM2.csv and estimates of α and β by least square method are given.



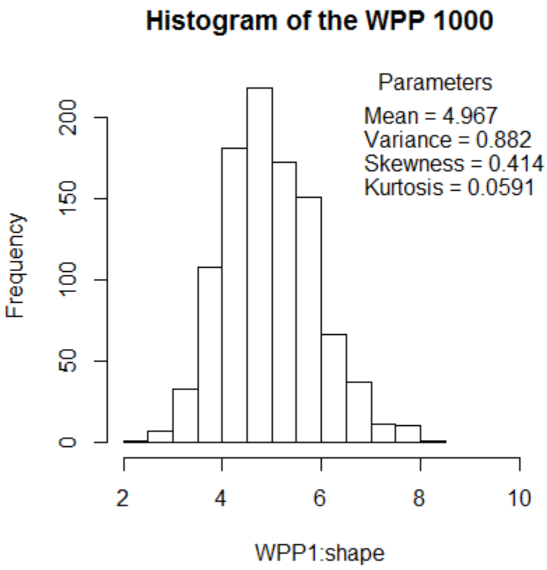


Figure 5.5: The histograms for estimation of parameters by the Weibull probability plot on 1000 random samples from dataSIM2.csv

5.4 Application of The Kolmogorov-Smirnov Test

The first function test.kstestfully compute the Kolmogorov-Smirnov statistic D for fully specified distribution and the second function test.kstescomposite compute the Kolmogorov-Smirnov statistic D for the composite hypothesis. Both functions use (3.6) and (3.7). The percentage points for the fully specified statistic are in Table 3.1. The percentage points for statistic with the composite hypothesis are in Table 3.2.

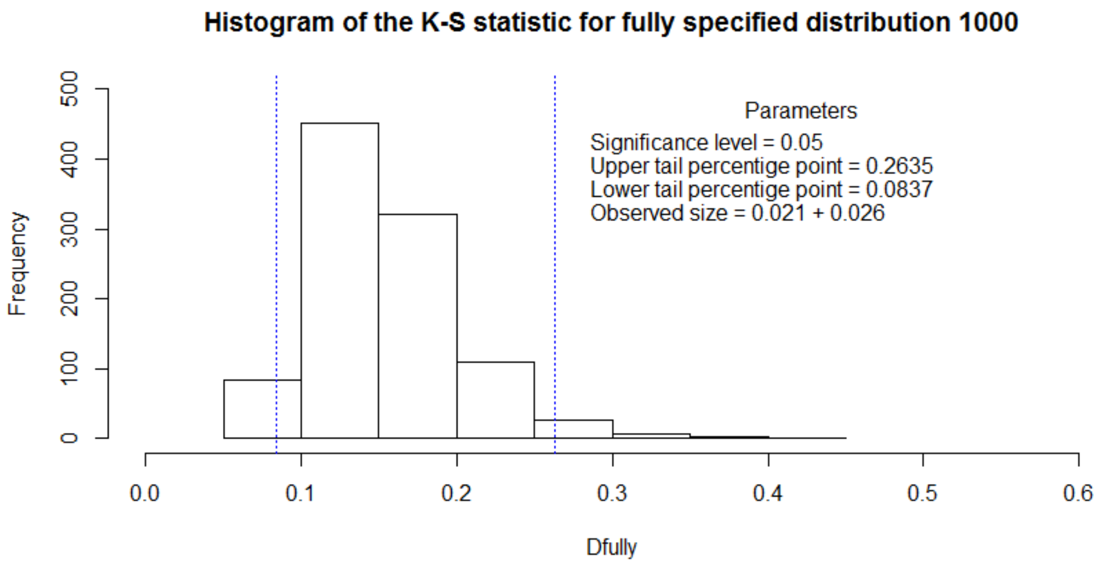


Figure 5.6: The histogram of the Kolmogorov-Smirnov statistic D for fully specified distribution with 1000 random samples with the significant level $\alpha = 0.05$ and the observed size of test equal to 0.047.

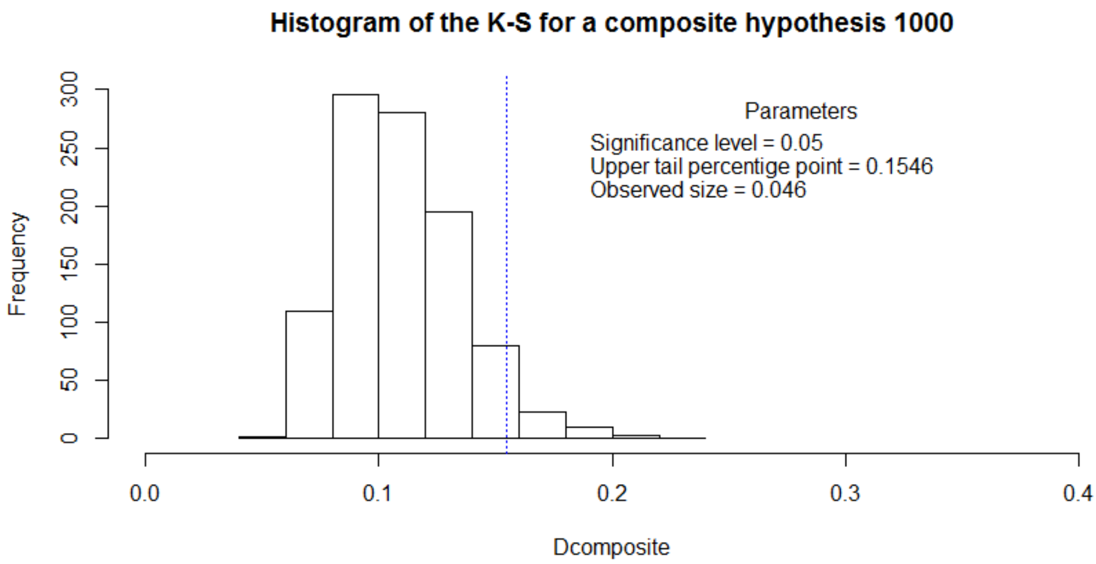


Figure 5.7: The histogram of the Kolmogorov-Smirnov statistic D for composite specified hypothesis with 1000 random samples with the significant level $\alpha = 0.05$ and the observed size of test equal to 0.046.

5.5 Application of The Anderson-Darling Test

The first function test.adtestfully compute the Anderson-Darling statistic A^2 for fully specified distribution and uses (3.10) and (3.11). The second function test.adtestcomposite compute the Anderson-Darling statistic A^2 for the composite hypothesis and uses (3.14) and (3.11). The percentage points for the fully specified statistic are in Table 3.1. The percentage points for statistic with the composite hypothesis are in Table 3.2.

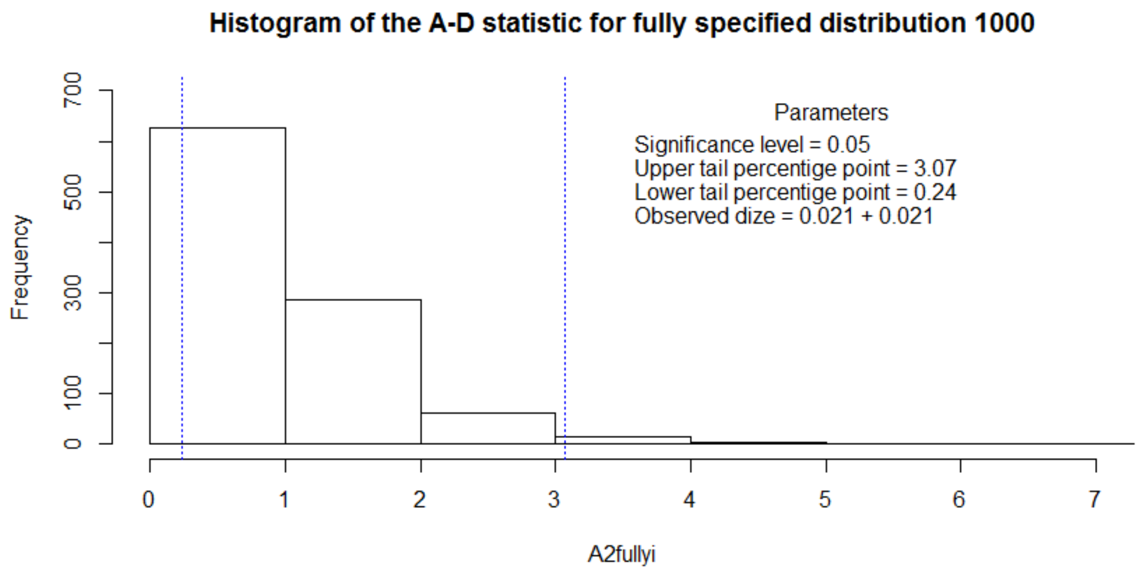


Figure 5.8: The histogram of the Anderson-Darling statistic A^2 for fully specified distribution with 1000 random samples with the significant level $\alpha = 0.05$ and the observed size of test equal to 0.042.

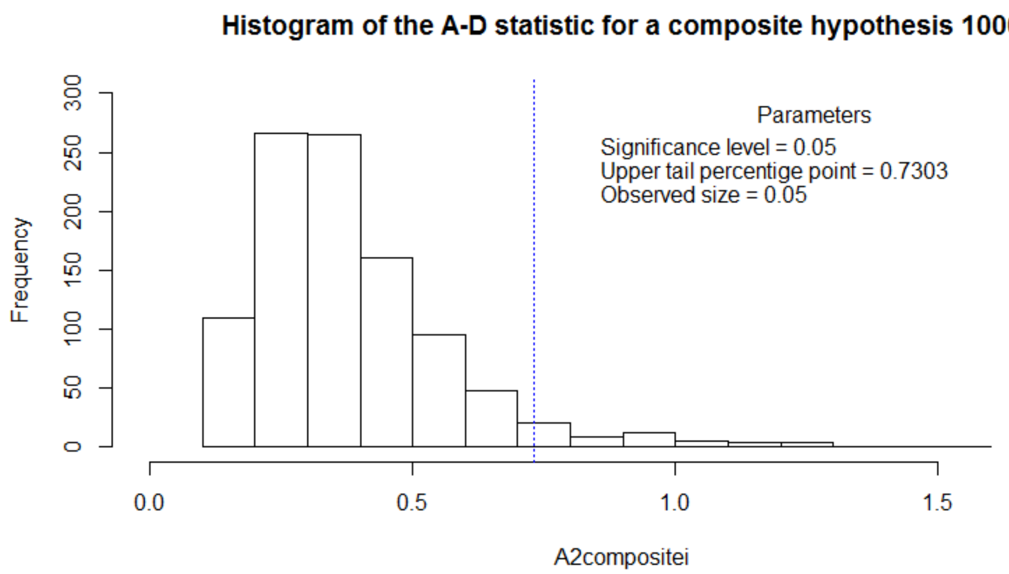


Figure 5.9: The histogram of the Anderson-Darling statistic A^2 for composite specified hypothesis with 1000 random samples with the significant level $\alpha = 0.05$ and the observed size of test equal to 0.05.

5.6 Application of MLE in ANOVA with two-parameter Weibull distribution

5.6.1 Application of model with constant shape parameter

On the beginning we must estimate the parameters of k random samples from table of data (for example dataSIM2.csv). It can be solved by function weibull.mleaB, which optimalizes the log-likelihood function of β (4.15) and computes the other parameters. Based on our experiences more precise estimation of parameters is obtained by finding the root of the equation (4.13). We must calculate the Fisher information matrix as (4.23) and a vector of the first derivatives (4.6), (4.7) and (4.8).

Then we can compute the tests - the score test (4.46), the Wald test (4.48) and the likelihood ratio test (4.50). For the tests with nuisance parameters we compute the Fisher information matrix and the score vectors as in the section (4.5) and we use the particular forms from the section (4.2).

5.6.2 Application of model with constant scale parameter

On the beginning we must estimate the parameters of k random samples from table of data (for example dataSIM2.csv). It can be solved by function of R multiroot, which estimates the root of the system of equations (4.25), (4.26) and (4.27), this roots are equal to the stationary points of the log-likelihood function and therefore to the maximum likelihood estimation of the parameters. Another option is maximization of log-likelihood function (4.24) by the R function One Dimensional Optimization (optim). We must calculate the Fisher information matrix as (4.35) and a vector of the first derivatives (4.25), (4.26) and (4.27).

Then we can compute the tests - the score test (4.46), the Wald test (4.48) and the likelihood ratio test (4.50). For the tests with nuisance parameters we compute the Fisher information matrix and the score vectors as in the section (4.5) and we use the particular forms from the section (4.3).

5.6.3 Example

Here we present an example of application of our results on simulated data, namely three random samples $\mathbf{X}_i$ from dataSIM2.csv with indexes $i = 7, 8, 9$.

On the beginning we testing the hypothesis that the three random samples have the two-parameter Weibull distribution with unknown parameters scale α and shape β.

Table 5.4: The values of the statistic tests of composite hypothesis for the significance level $\alpha = 0.05$ for the random sample $\mathbf{X}_i$ for $i = 7, 8, 9$ from dataSIM2.csv.

Random sample	D	A^2
$\mathbf{X}_7$	0.0901981	0.204213
$\mathbf{X}_8$	0.1368182	0.4560774
$\mathbf{X}_9$	0.1115631	0.3719207
The upper percentile point	0.1546403	0.730332

The formulas for statistic test and the Table 3.2 with the upper percentile point are in the section Tests of composite hypothesis based on EDF statistic (3.2.4).

From the Table 5.4 we do not reject the null hypothesis, that the three random samples are from the two-parametric Weibull distribution at the significant level $\alpha = 0.05$.

We can assess the quality of goodness of fit by scanning the Weibull probability plot in Figure 5.6.3.

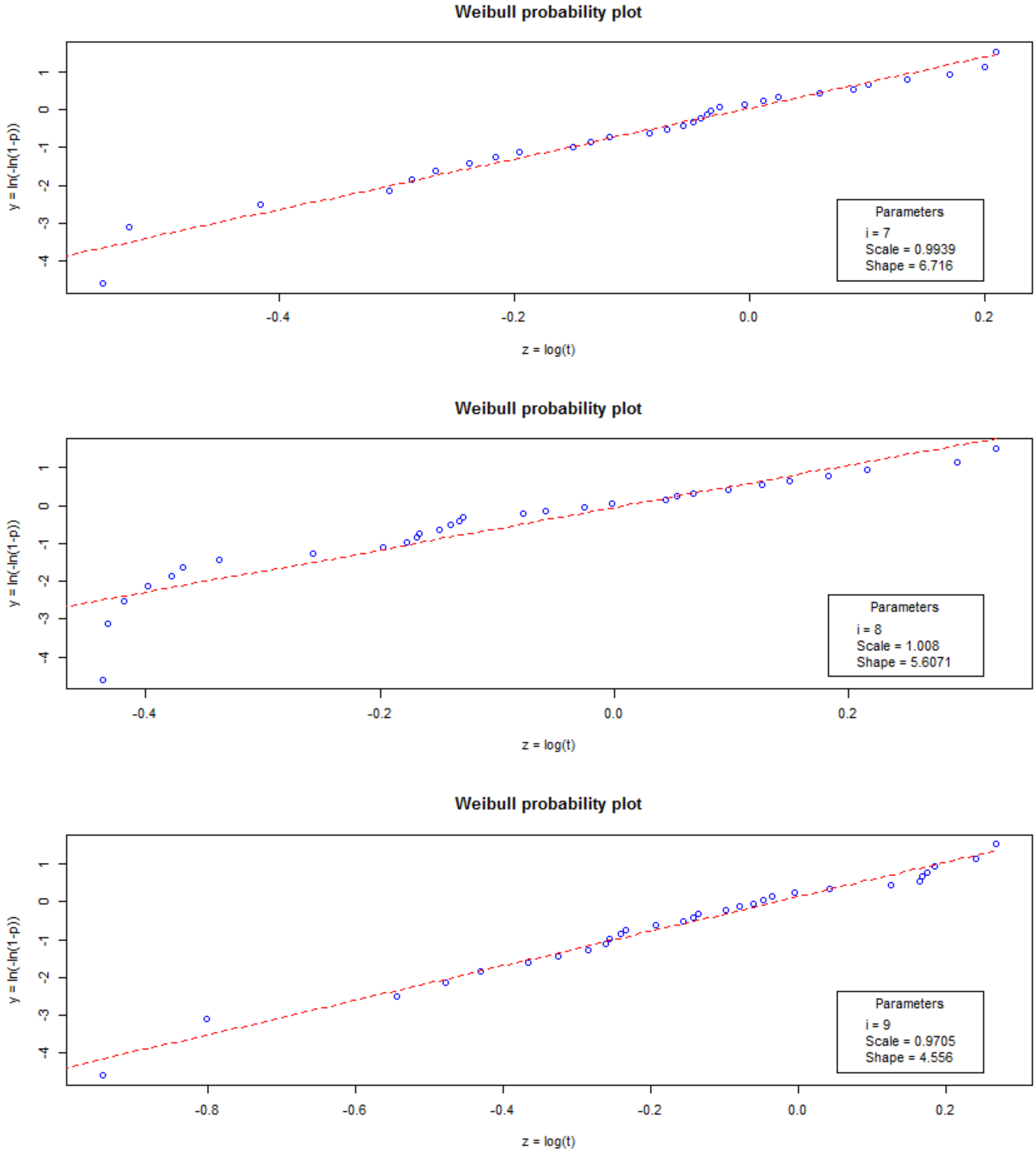


Figure 5.10: The Weibull probability plots for the random samples $\mathbf{X}_i$ for $i = 7, 8, 9$.

First we consider the test of equality of the scale parameter under the assumption of constant scale parameter.

Table 5.5: The model with constant shape parameter for the three random samples $\mathbf{X}_i$ for $i = 7, 8, 9$ $(k = 3)$ from dataSIM2.csv with condition $\delta_1 = 0$.

Parameters	ξ	δ_2	δ_3	β
$\hat{\boldsymbol{\theta}}_n$	0.94859535	-0.01742858	0.05823429	5.004600
$\tilde{\boldsymbol{\theta}}_n$	0.9635591	0.0	0.0	4.9510
Statistic of the test	S	W	LR	$\chi^2_2\,(0.05)$
Value of the statistic of the test	2.667692	2.454278	2.554172	5.991465

The null hypothesis for the tests has form

$$H_0 : \left(\hat{\delta}_2,\, \hat{\delta}_3\right)^T = (0,\, 0)^T .$$

The null hypothesis is rejected if the value of the test statistic is greater or equal to $\chi^2_2\,(0.05)$ at the significance level 0.05. All statistic are smaller than $\chi^2_2\,(0.05) = 5.991465$ which means that we do not reject the null hypothesis at the significance level 0.05.

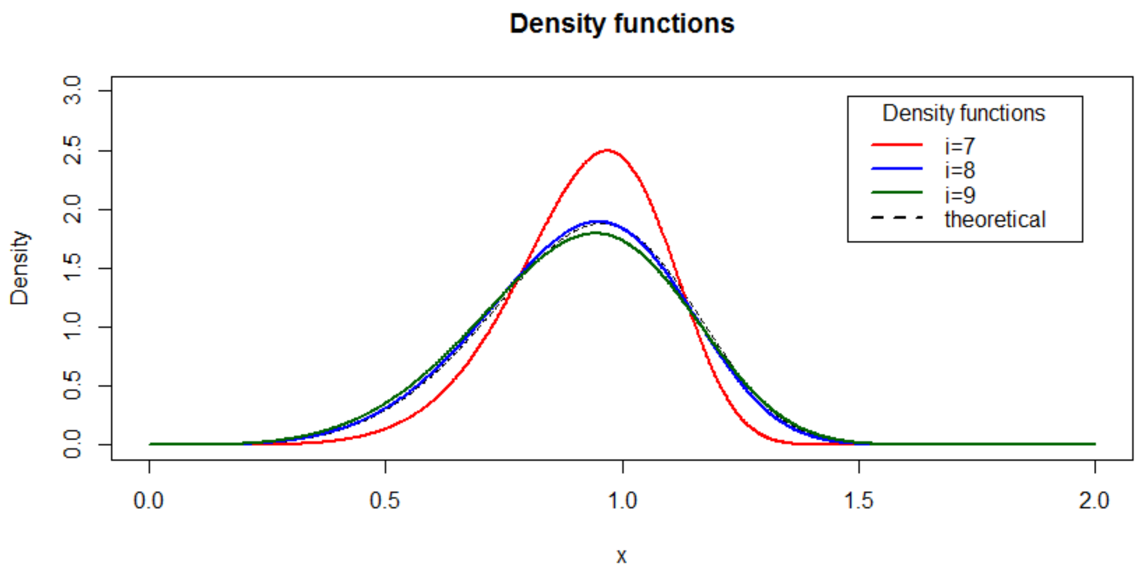


Figure 5.11: The graphs of the density functions with parameters from Table 5.5

In the Figure 5.6.3 we can see three density functions with the same parameter scale α and one density function with parameters $\alpha = 1$ and $\beta = 5$ which correspond to the parameters of the simulation of random samples from dataSIM2.csv. Next we consider the test of equality of the shape parameter under the assumption of constant scale parameter

Table 5.6: The model with constant scale parameter for the three random samples $\mathbf{X}_i$ for $i = 7, 8, 9$ $(k = 3)$ from dataSIM2.csv with condition $\epsilon_1 = 0$.

Parameters	α	ϵ_2	ϵ_3	ν
$\hat{\boldsymbol{\theta}}_n$	0.9728761	−0.0053723	0.2103764	5.2763408
$\tilde{\boldsymbol{\theta}}_n$	0.9911211	0.0	0.0	5.2737197
Statistic of the test	S	W	LR	$\chi^2_2\,(0.05)$
Value of the statistic of the test	3.351164	3.253781	3.469002	9.487729

The null hypothesis for the tests has form

$$H_0 : (\hat{\epsilon_2},\, \hat{\epsilon_3})^T = (0,\, 0)^T \,.$$

The null hypothesis is rejected if the value of the test statistic is greater or equal to $\chi^2_2\,(0.05)$ at the significance level 0.05. All statistic are smaller than $\chi^2_2\,(0.05) = 9.487729$ which means that we do not reject the null hypothesis at the significance level 0.05.

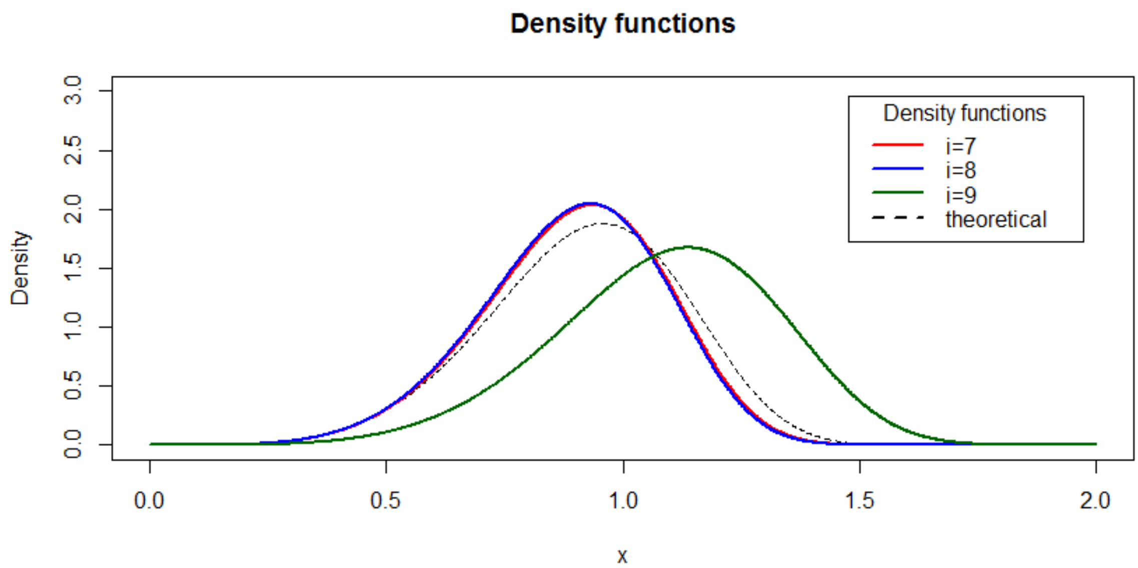


Figure 5.12: The graphs of the density functions with parameters from Table 5.6

In the Figure 5.6.3 we can see three density functions with the same parameter shape β and one density function with parameters $\alpha = 1$ and $\beta = 5$ which correspond to the parameters of the simulation of random samples from dataSIM2.csv.

We do not reject the both hypothesis for the tests of equality of parameter at the significance level $0, 05$. From this we can deduce that the parameters are constant for all three random samples $\mathbf{X}_i$ for $i = 7, 8, 9$. Because the random samples were generated with the same parameters, we know that the hypothesis about constant parameters is correct.

Conclusion

This Master's thesis Model with Weibull responses introduced a description of the properties of Weibull distribution and several parameter estimation methods.

The theory that was used later in the next chapters was introduced in chapter 0.

Chapter 1 presents the general characteristics of Weibull distribution and specifically for the two-parameter Weibull distribution.

In Chapter 2 the estimation of parameters are presented. All four different way of methods of quantiles derived and are coded in program Rkod_MT_Konecna.R. The main result of this section is conclusion that the general case of method of quantiles for $p_1 = 0.23875930$, $p_2 = 0.92656148$ is the best in the sense of minimal determinant of the asymptotic variance matrix. This was demonstrated by theoretically, by table and by simulations.

In Section Method of Maximum Likelihood the regularity of a system was defined. A proof that the two-parameter Weibull distribution is a regular system was provided, then the system of likelihood equations and the fisher information matrix were derived and used for the estimation of the parameters of the distribution

The results of the graphical method of estimation of parameters of the Weibull distribution were provided in a form of a probability plot. In this method we applied the least squares estimation. The code for this method can be found in the file Rkod_MT_Konecna.R.

Chapter 3 described the tests of hypotheses that a random sample comes from a Weibull distribution. The goodness of fit of the Weibull distribution can be assessed by the test of χ^2-type and a test based on EDF statistics, namely the Kolmogorov-Smirnov statistic and the Anderson-Darling statistics. In the subsection Anderson-Darling Statistic we derived the form of statistic A^2 for hypothesis that random sample has the two-parameter Weibull distribution. In the subsections Test of Fully Specified Hypothesis Based on EDF Statistic and Test of Composite Hypothesis Based on EDF Statistic we can see the table with modified form of statistic and the percentage point.

In the Chapter 4 the derivation of parameter estimation methods in the one-way ANOVA type models with Weibull distribution was presented. Relations for the model with constant scale parameter α, constant shape parameter β and the model with both parameters constant were derived. Also the tests with nuisance parameters are included, namely the score test, the Wald test, and the likelihood ratio test.

Chapter 5 deals with the applications of the methods presented in Chapters 2, 3 and 4. A comparison of the different methods are demonstrated by graphs, histograms and tables. The methods are programmed in R and GAMS software. The functionality and properties of each method are verified on three sets of simulated data.

In the end of this chapter as an example three random samples from dataSIM2.csv are analysed. In this example the methods from the file Rkod_MT_Konecna.R were applied.

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List of Abbreviations and Symbols

Symbol	Name
A^2	The Anderson-Darling statistic
$\mathbf{B}$	The matrix of partial derivatives
$\mathrm{Cov}\,(x)$	The covariance of x
D	The Kolmogor-Smirnov statistic
E_i	The frequency of a class
$f\,(x, \boldsymbol{\theta}))$	The density function
$F\,(x, \boldsymbol{\theta}))$	The distribution function
$F_0\,(x, \boldsymbol{\theta}))$	The distribution function
$\hat{F}_n\,(x)$	The empirical distribution function
$\hat{F}_n^*\,(x)$	The modified distribution function
h_i	The real-value function
$\mathbf{H}\,(x, \boldsymbol{\theta})$	The Hessian matrix
i	index
$\mathbf{I}_n\,(\boldsymbol{\theta})$	The Fisher information matrix
j	index
$L\,(\boldsymbol{\theta})$	The likelihood function
$L_i\,(\boldsymbol{\theta})$	The likelihood element
$\mathcal{L}\,(\boldsymbol{\theta})$	The log-likelihood function
$LR\,(\boldsymbol{\theta})$	The likelihood ration test
m	Number of parameters in $\boldsymbol{\theta}$
n	The sample size
p	The quantile
$Q\,(p)$	The quantile function
Q_i	The theoretical quantile
$\hat{Q}\,(p)$	The estimation of quantile
$S\,(\boldsymbol{\theta})$	The score test
t_i	The border point of interval
$U\,(\boldsymbol{\theta})$, $U_i\,(\boldsymbol{\theta})$	The score
$\mathrm{Var}\,(x)$	The variance of x
$W\,(\boldsymbol{\theta})$	The Wald test
X_i	The event
$X_1, \ldots, X_n$	The random sample of size n
$\mathbf{X}$	The random sample
$\mathbf{X}_i$	The i-th random sample
$X_{(1)}, \ldots, X_{(n)}$	The ordered random sample
$X_{(1),n}, \ldots, X_{(n),n}$	The ordered random sample of size n
$X_{p_i,n}$	The sample quantile

Symbol	Name
α	The significant level
α, α_i	The parameter scale
$\hat{\alpha}$	The estimation of the parameter scale
β, β_i	The parameter shape
$\hat{\beta}$	The estimation of the parameter shape
$\Gamma(z)$	The complete Gamma function
δ_i	The parameter
ϵ_i	The parameter
$\boldsymbol{\theta}$	The general vector of parameters
μ	The finite σ-measure
μ	The expected value
ν	The parameter
ξ	The parameter
$\rho_{ij}\sigma_i\sigma_j$	The covariance
σ^2	The variance
σ_i^2	The variance of parameter θ_i
$\mathbf{\Sigma}$	The covariance matrix
τ	The vector of parameters
ψ	The vector of parameters
$\psi^{(0)}(z)$	The polygamma function
Ω	The parameter space
ω	The part of the parametric space

Abbreviation	Sense
MLE	Maximum likelihood estimation
EDF	The empirical distribution function

Electronic Appendix Index

1. **dataSIM.csv** - the simulated data with the number of samples $m = 1000$, sample size $n = 30$, the parameters $\alpha = 1$ and $\beta = 50$ and with results of methods.
2. **dataSIM2.csv** - the simulated data with the number of samples $m = 1000$, sample size $n = 30$, the parameters $\alpha = 1$ and $\beta = 5$ and with results of methods.
3. **Rkod_MT_Konecna.R** - the file with code in the software R.